ELECTRIC CIRCUITS

AND FIELDS

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FIRST EDITION
THIRD IMPRESSIC

PREFACE

In the following pages are given, from an engineering point of view, (1) a description of the more important effects commonly described as electric and magnetic phenomena; (2) a statement both in words and in mathematical formulas of the fundamental laws or principles in accordance with which these phenomena have been found to be related; and (3) the application of these principles to some of the simpler problems that arise in connection, with the generation, transmission, and utilization of electric energy.

Particular emphasis is laid upon exact quantitative statements of the fundamental principles. Both safety and economy demand that the engineer be able to answer, not only "how," but also "how much." To this end, the student of engineering should be taught to analyze, not only qualitatively, but also quantitatively, each problem that may be presented to him.

This book is designed for students who have had college courses in differential and integral calculus and in general physics. In the development of the engineering aspects of electricity and magnetism here given, much of the information that the student has acquired in his course in physics is reviewed in detail, but not always in the sequence in which he acquired his first knowledge of this branch of physics. In particular, this book begins with a detailed analysis of electric circuits, with only incidental reference to electric and magnetic fields. Following the discussion of electric circuits the theory of electric and magnetic fields is developed in detail.

Most of the simpler formulas used by scientists and engineers are special cases of more general relations, and these special formulas are applicable only under certain specific conditions. One of the most common causes of confusion on the part of the beginner arises from his attempt to apply such special formulas to cases to which they are not applicable. This is due in part to the failure in many textbooks to state the limitations of such

formulas. Particular care is therefore taken in these pages to state specifically the exact conditions under which each formula is applicable.

As in two former books written by the senior author—*Principles of Electrical Engineering* and *Electricity and Magnetism for Engineers*—this volume uses as the foundations of all analyses the *useful effects* of what is termed electricity and magnetism—effects such as heat, light, mechanical motion, and chemical reactions; in short, those things which are directly observable. The entities such as electric charge, electric current, lines of magnetic force, and the mathematical relationships connecting them will be considered as useful tools, as means to ends—practical and necessary ends of engineering development, construction, operation, and maintenance.

The text should preferably precede or else be given concurrently with texts on measurements and machinery. An effort has been made to cover the fundamental principles of electrical engineering without disturbing the continuity by the inclusion of detailed applications, which in the experience of the authors are better given separately.

Where there has been a choice of methods of approach to a given problem, that method has been chosen which is the most practical with respect to engineering application, always, of course, with due regard to rigor. For example, magnetic flux is defined in terms of the electromotive force produced by a change of flux (the practical reason for the concept of flux) rather than as the flux from a magnetic pole, which latter is of much less utility to the engineer and which becomes extremely confusing in dealing with the phenomena *inside* magnetic materials.

During the past five years advances in the fields of radio, electronics, and ultra-high-frequency techniques have been accelerated. There is an increasing demand that students who graduate with a bachelor's degree in electrical engineering have information concerning the fundamentals underlying the techniques of these fields of investigation. This demand is reflected even in elementary courses in electrical engineering. Therefore, in this book problems of the electrostatic and electromagnetic fields have been described in terms of vector concepts, and chapters on the fundamentals of electronics and electromagnetic radiation are included.

The rationalized mks system of units is used throughout the book. An mks system was chosen because the authors believe that it combines effectively the advantages of the metric system and the advantages of the units now commonly used by electrical engineers—units such as the volt, ampere, ohm, henry, and farad. The *rationalized* mks system was chosen for more arbitrary reasons; the authors find that the factor 4π appears explicitly in this system in those relations in which the authors are least interested. Thus the factor 4π appears in Coulomb's law and in Ampère's law; although the phenomena which these laws represent are fundamental, they are not commonly used in electrical engineering computations. The factor 4π does not appear in the expression for magnetomotive force and it does not appear in Maxwell's equations; these forms are often used in electrical engineering computations.

Although the scope of the material covered by this book is extensive, an attempt has been made to choose the subject matter with a view to providing a framework upon which any teacher may build according to his own taste. Many subjects are not discussed in detail in the body of the text. An instructor may bring subjects not so included into his course by one of two procedures, both of which have been found to be extremely useful in the authors' experience: (1) Students may be required to look up references listed at the end of each chapter, and these, in turn, will lead him to an extended bibliographical study; (2) many subjects normally analyzed in detail within a textbook are stated as problems at the end of the chapters.

The authors have two purposes in arranging the book in this manner: (1) to permit each instructor who uses the book to emphasize those aspects of the fundamentals of electrical engineering which he thinks should be emphasized; (2) to encourage the student relatively early in his college career to learn to refer to the works of many authors and to technical journals. It is the authors' opinion that the student is most likely to progress quickly in his advanced work if he has acquired the habit of using a library well during the sophomore or junior year.

There are, of course, differences of opinion among engineers concerning the scope and content of the academic work that should be required for a degree in electrical engineering. Regardless of these differences of opinion, the authors believe that it is

important to instruct students of electrical engineering in such a manner that they may acquire (1) a general qualitative theory correlating the fundamental phenomena encountered in electrical engineering practice; (2) simple mathematical rules by which they may calculate quantitatively the effects produced by the connection and use of simple electrical devices; (3) an attitude toward problems in any field that utilizes the effectiveness of what is called the *scientific method*. This book is formulated and arranged with these three purposes in mind.

In conclusion the authors wish to point out that this book is an outgrowth of the course of lectures based on the senior author's *Electricity and Magnetism for Engineers,* which has been used for many years as a textbook in the junior course in electrical engineering given at the Moore School of Electrical Engineering, University of Pennsylvania. During this period this course has been taught not only by the two authors, but also by Professors Charles Weyl, Knox McIlwain, and Irven A. Travis, all of whom have contributed to the ideas and methods of presentation given in the present volume. The authors are indebted also to Solomon Charp for critical comments based upon his reading of the proof of this book.

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October, 1943.

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PART 1

Electric-circuit Theory

FUNDAMENTAL CONCEPTS OF ELECTRIC-CIRCUIT THEORY

1.01. Introduction.—Problems in electrical engineering can be conveniently divided into two classes:

1. *Electric-circuit* problems, which involve the investigation of the measurable effects of electric currents, either constant or varying with time, that are confined to their sources (such as batteries and generators) and to resistors, inductors, condensers, and the wires used to interconnect them.

2. *Electric- and electromagnetic-field* problems, in which the relevant quantities such as the electric- and magnetic-field intensities and electric-charge density must be specified throughout regions that are external to the pieces of apparatus used to produce the observed effects.

The elementary mathematical theory of the electric and electromagnetic fields is more complicated than the elementary theory of electric circuits. Furthermore, the laboratory experiments performed by students of electrical engineering concurrently with the study of electricity and magnetism are usually experiments on electric circuits. Therefore the first part of this book is devoted to the analysis of electric circuits and the second part is devoted to the analysis of electric and electromagnetic fields.

It is the purpose of this chapter to define the fundamental concepts of electric-circuit theory. These fundamental concepts are applicable to all circuits in which the mathematical relations among the relevant physical quantities (such as electric potential difference, electric current, resistance, and capacitance) are linear,

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irrespective of whether the currents are constant or are varying with time.

The fundamental concepts defined and illustrated in Chap. I are applied in Chap. II to the particularly simple cases of electric circuits in which the currents are constant; this is called *direct-current circuit theory*.   The more complicated theory of circuits in which the currents vary with time is developed in subsequent chapters of Part 1.

1.02. Units and Dimensions.—The quantities (such as electric current and resistance) that are defined in subsequent sections of this chapter are specified quantitatively in terms of agreed-upon *units*.   There are several commonly used systems of units in physics.   In each system, however, all quantities can be expressed in terms of the agreed-upon units of *mass, length, time,* and two other quantities—for example, electric charge and temperature.   Thus five quantities in each system are of peculiar importance; any other quantity expressed in terms of two or more of the five is said to have the *dimensions* so expressed.   For example, the quantity *acceleration* is said to have the *dimensions:*

$$(\text{Acceleration}) = (\text{length})^1 \ (\text{time})^{-2}$$

The student will recall other examples from his knowledge of elementary physics.

Useful mathematical relations in physics have a peculiar property: *all the additive terms of an equation have the same dimensions*.   This is called the *principle of dimensional homogeneity*.   It is often useful as a check against errors in mathematical calculations, and it will be so used in this book.   As an example, consider the relation among the velocity v of a freely falling body near the surface of the earth, the initial velocity v_0 of the body, the acceleration of gravity g, and the time t between the instants for which the velocities are v and v_0

$$v = v_0 + gt$$

in which each of the three terms has the dimensions $(\text{length})^1$ $(\text{time})^{-1}$ [see also Eq. (1.13)].

Temperature need not be used explicitly in the mathematical formulation of most problems of electrical engineering.   Therefore four fundamental quantities—mass, length, time, and one other—are required for a dimensional system for electricity and

magnetism; and all other quantities used in the theory can be expressed dimensionally in terms of these four.   The fourth quantity chosen to complete the dimensional system used in this book is *electric charge*.

There remains the problem of choosing a *unit of measurement* for each of the four quantities—length, mass, time, and electric charge.   The unit system used in this book is the system internationally agreed upon by physicists and engineers as one of the standard systems after Jan. 1, 1940.   The fundamental units of this system are

Unit of length................................. meter
Unit of mass.................................. kilogram
Unit of time.................................. second
Unit of electricity........................... coulomb

If temperature is considered as a fifth fundamental quantity, the standard unit for it is the degree centigrade.

It is here assumed that the reader has studied elementary courses in mechanics and electricity and magnetism, and that he is familiar with these units and the manner in which measurements are expressed numerically in terms of them and their multiples and submultiples.   For a detailed specification of these several units the reader is referred to any modern treatise on physics.[1]

The complete system of units based upon the meter, kilogram, second, and coulomb will be referred to as the mks or mksq system of units and will be used throughout this book.

Some of the more common derived mksq units are

Unit of force................................. newton
Unit of work or energy........................ joule
Unit of power................................. watt
Unit of electric current...................... ampere
Unit of electric potential difference or emf......... volt

A newton is the force required to accelerate 1 kilogram at the rate of 1 meter per second per second.   A joule is the work done by a force of 1 newton when the rigid body on which this force acts is displaced linearly one meter.   A watt is the power, *i.e.*, rate of doing work, that corresponds to 1 joule per second.

[1] Superior numerals refer to items in the References at the end of the chapter.

An ampere is the current of electricity that corresponds to a flow of 1 coulomb per second. A volt is the work required to displace 1 coulomb of electricity 1 meter against an opposing force of 1 newton.

In order to get the "feel" of the magnitudes of the several derived units just defined, the reader should note the following facts. A newton is approximately equal to the weight of a big apple, or more specifically the force corresponding to a weight of 0.225 pound. A joule is approximately equal to the work required to lift a big apple through a vertical distance of 3 feet, or the energy required to heat a pint of water approximately $\frac{1}{1000}$ degree Fahrenheit. A watt is approximately one-thirtieth of the mechanical power output of an ordinary 8-inch electric fan, or one-third of the electric power input to an ordinary electric clock. An ampere is approximately one-thirtieth of the current output of a new No. 6 dry cell when it is short-circuited through an ammeter. A volt is approximately two-thirds the difference of electric potential between the terminals of a new No. 6 dry cell, or one-one hundredth of the difference of electric potential between the terminals of a household electric circuit.

Having chosen four dimensional quantities and units for them, it is of course necessary to set up and maintain standards of these units to which *secondary standards* can be compared; the secondary standards can then be used to calibrate instruments used throughout the world. Such standards, in the United States, are constructed and maintained by the United States Bureau of Standards, Department of Commerce, Washington, D.C. The student is referred to the publications[2] of the bureau for a discussion of this work, with which the practicing engineer is seldom concerned.

Since the centimeter-gram-second systems—both electrostatic and electromagnetic—are still commonly used, a table of relations among units of electric charge, electric current, electric potential difference, and electric resistance in the two cgs systems and in the mks system is appended to this chapter (page 41). Tables of conversion factors for other units can be found in most electrical engineering handbooks.[3]

1.03. Review of Elementary Atomic Theory.[4]—It is the purpose of this section to describe briefly a theory of the constitution of matter that can be used as a physical basis for the discussion

of electric-circuit theory. It is assumed that the student has studied in his physics courses the elementary experiments upon which this simple theory is based.

All matter is assumed to consist of combinations of *atoms* of 92 *elements.* Each atom in turn is assumed to comprise a relatively stable *nucleus* of *protons* and *neutrons* having a net positive charge, and a group of external *electrons* bound in some way to the nucleus to form an electrically neutral atom, *i.e.*, an atom having a net electric charge of zero. Electrons, protons, and neutrons are identified by their electric charges and by their masses. The charges and masses of these entities are listed below; note that the ratio of the mass of an electron to the mass of either a proton or a neutron is approximately $1/1846$. Thus the mass of an atom resides chiefly in the nucleus, and

Entity	Mass, kg.	Charge, coulombs
Proton........	1.66×10^{-27}	1.59×10^{-19}
Neutron......	1.66×10^{-27}	0
Electron......	9.035×10^{-31}	-1.59×10^{-19}

it can be conveniently though approximately expressed in terms of the integral number of protons and neutrons in the nucleus of the atom. Such an expression of the mass of an atom is called the *mass number*. The mass numbers of the atoms of the 92 elements are from 1 (for hydrogen) to 238 (for uranium). The integral number of protonic charges of the nucleus or the equal integral number of external electrons associated with the nucleus is the *atomic number*. The atomic numbers specify the elements, from 1 (for hydrogen) to 92 (for uranium). Atoms having equal atomic numbers but different mass numbers are *isotopes* of the same element. Thus most atoms of hydrogen have a mass number of 1, but a few atoms have mass number 2 and are called atoms of *deuterium* or *heavy hydrogen;* the atomic number in both cases is 1.

In accordance with this picture each atom in its normal state contains equal amounts of positive and negative electricity.

* Note that recent experiments in the field of nuclear physics have led to the production of small amounts of elements not heretofore found on the earth.[5]

This state of affairs is usually described by saying that the *net* electric charge in a neutral portion of matter is zero. At distances from a neutral atom large compared with its linear dimensions no appreciable *electrostatic* effects are produced, *i.e.*, there exist no effects that are attributable to the *positions* of the electric charges in the atom, although there may exist effects attributable to the *motions* of these charges.

1.04. Conductors and Insulators.—In solids the nuclei of atoms are assumed to remain essentially stationary in a fixed pattern. The electrons on the other hand are thought of as being in motion, but bound to their respective nuclei in the sense that energy is required to remove any external electron from the influence of its nucleus. The amount of energy required varies from one electron to another in a particular atom, and it depends also on the kind of atom. Thus it is consistent with this theory to assume that, under the influence of an external source of energy, electrons that are "loosely" bound to the nuclei of the atoms in a solid may move from atom to atom, resulting in a drift of negative electricity through the solid as a whole. Such a drift is called a *conduction current* of electricity, and the electrons whose motions give rise to this drift are called *free* electrons. A solid, such as a metal or carbon, in which such a drift is produced by relatively weak external forces is called a *conductor*. A solid, such as glass, dry fabrics, paper, sulphur, etc., in which a measurable conduction current can be produced, without rupturing the material, only by a relatively strong external force, is called an *insulator*, or *dielectric*.

In liquids and gases the observed experimental facts, when interpreted on the electron theory, justify the assumption that a flow of electricity may take place, electrons from atoms moving through any specified area in one sense, and the remainders of such atoms, called *positive ions*, moving in the opposite sense. When a conduction current flows through an inorganic solution a chemical reaction usually takes place at the electrode at which the current enters or leaves the liquid. Such liquids are called *electrolytes*, and the flow of electricity through them is commonly referred to as *electrolytic conduction* as distinguished from the kind of flow which takes place in a solid, which is usually referred to as *metallic conduction*, even though the solid itself is not a metal (for example, carbon).

In general, the net rate at which the negative electricity in any substance (solid, liquid, or gaseous) moves in a given sense through a given area *plus* the net rate at which the positive electricity moves in the opposite sense through this area is taken as the measure of the conduction current through this area, and the *positive sense* of the current is always taken as the sense in which the positive electricity moves, or, what amounts to the same thing, the sense *opposite* to that in which the negative electricity moves.

1.05. Magnets and Magnetic Substances.—In the electron theory a *magnet,* of which an ordinary compass needle is the simplest example, is considered as a substance in which some of the electrons have a spinning or rotary motion about axes that are more or less parallel to each other.   When such a substance is demagnetized, or unmagnetized, these axes have random directions.   The process of magnetization consists in turning these axes, by some external force, into more or less parallel directions.   The more highly magnetized a body is, the more nearly parallel these axes become.   From this point of view there is of course a limit to the degree to which a body may be magnetized, which is in accord with experimental fact.   A substance, such as iron, nickel, cobalt, and certain special alloys, that is readily magnetized is called a *magnetic substance,* or more specifically a *ferromagnetic* substance.   Most elements have magnetic properties similar in kind to those of the ferromagnetic substances, although the magnetic effects are so weak that they are difficult to measure; these substances are called *paramagnetic.* A few elements, such as bismuth, exhibit *diamagnetic* properties, *i.e.,* the magnetic effects of their atoms tend to oppose the magnetic effects produced by the presence of other magnets near them.   These differences in magnetic properties of various substances are discussed in detail in Chap. XIV.

1.06. Difference of Electric Potential or Voltage.—In order to correlate the observed experimental facts with the electron theory it is necessary to recognize several different types of forces that may act upon electrons and protons, or upon groups of such particles of electricity, usually referred to as electric *charges.*

First of all, there exists between any two like charges a force of repulsion, and between unlike charges a force of attraction, which depend only upon the relative *positions* of these charges

and the quantity of electricity in each. Such forces will be referred to as *electropositional* forces. A second type of force is that which is exerted upon a *moving* charge in the vicinity of other moving charges. Such forces, which depend not only upon the positions but also upon both the velocities and the accelerations of the moving charges, as well as upon their amounts, will be referred to as *electromotional* forces.* Since a magnet is considered as an aggregation of charges that have a spinning or rotary motion, the force exerted by a magnet on the moving charges that constitute an electric current may always be considered as an electromotional force.

A third type of force is the force that opposes the motion of an electric charge through a material medium and that manifests itself by the production of heat energy within the medium. Such forces are analogous to ordinary frictional forces and will be referred to as *electrofrictional* forces. A fourth type of force is that which is exerted upon the protons and electrons at the surface of separation of two different conductors, such as a metal and an electrolyte, and which tends to keep these charges *separated*. Such a force may be referred to as an *electrochemical* force.

When a particle of electricity moves from any point P_1 to another point P_2 each of the various forces acting upon it does work, or work is done against this force. As a rule one of the forces that does work or has work done against it is the *electropositional* force attributable to the presence in the vicinity or elsewhere of other electric charges. Imagine a particle that has an infinitesimal *positive* charge dQ to be displaced from P_1 to P_2 along any specified path in an infinitesimal† time dt measured from some instant of time t. Let dW be the amount of work done on this charge dQ *by the electropositional forces* produced on it by all the other charges in space. The quotient

$$(1.01) \qquad\qquad \frac{dW}{dQ} \qquad \text{joules per coulomb}$$

* The electromotional force on a given charge consists in general of two parts; one part depends on the *position* of this charge (see Sec. 12.03); the other part is perpendicular to the first and depends upon the *velocity* of this charge (see Sec. 14.07).

† Or in an interval of time so short that no change takes place in the distribution of the surrounding charges.

is *defined* as the *difference of electric potential* between the two points P_1 and P_2 at time t. This quantity has the extremely important property that for any two specified points fixed with respect to an unvarying distribution of electric charges it has *one and only one value,* which value is *independent of the path* along which the charge dQ is displaced.

The quantity dW in Eq. (1.01) is a *positive* number when work is done *by* the electropositive force *on* the charge dQ and is a *negative* number when work is done *by* the charge dQ *against* this force. Hence the difference of potential V_{12} defined by Eq. (1.01) may be either a positive or a negative number.

For a given distribution of electric charges the *difference* of electric potential V_{12} between two specified points P_1 and P_2 may therefore be considered as the difference between a quantity V_1 that has one and only one value at P_1 and a quantity V_2 that has one and only one value at P_2, *viz.*,

$$(1.02) \qquad\qquad V_{12} = V_1 - V_2 \qquad \text{joules per coulomb}$$

The quantities V_1 and V_2 are called the electric *potentials** at the two points P_1 and P_2, as distinguished from the electric *potential difference between* these two points. The unit of electric potential difference and of electric potential (joule per coulomb) is called the *volt.* Either of the two quantities V_1 and V_2 may be a positive or negative number. When V_1 is algebraically greater than V_2 then the point P_1 is said to be at the higher potential, and the positive number V_{12} is called the *drop of electric potential* from P_1 to P_2. A negative value of V_{12} means a *negative drop* of potential from P_1 to P_2 or a *positive rise* of potential in the opposite sense, *i.e.,* from P_2 to P_1.

In terms of the difference of electric potential V_{12} between two equipotential surfaces, the work done by the electropositional forces when dQ' units of positive electricity flow from a specified area in an equipotential surface No. 1 to a specified area in an

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* The choice of the "absolute" value of the potential at any point is arbitrary to within a constant, since

$$(V_1 + C) - (V_2 + C) \qquad V_1 - V_2$$

where C is any number that is independent of the location of the point under consideration and also independent of time,

equipotential surface No. 2, in time dt, is

$$(1.03) \qquad\qquad dW' = V_{12} \, dQ' \qquad\qquad \text{joules}$$

Similarly, when dQ'' units of negative electricity flow in the *opposite* direction along this *same* path, the work done by the electropositional forces on this negative charge in time dt is

$$(1.04) \qquad\qquad dW'' = V_{12} \, dQ'' \qquad\qquad \text{joules}$$

The total work done in the time interval dt by the electropositional forces on the positive and negative charges in this path is then

$$(1.05) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{joules}$$

where dQ is the arithmetic sum of dQ' and dQ''. The rate at which this work dW is done is then

$$(1.06) \qquad\qquad \overline{dt} \qquad\qquad \overline{dt} \qquad\qquad \text{watts}$$

In this expression dQ/dt is the arithmetic sum of the rate at which positive electricity leaves the specified area in surface No. 1 (or enters the specified area in No. 2) and the rate at which negative electricity enters the specified area in No. 1 (or leaves the specified area in No. 2). This sum is by definition the electric *current* that leaves surface No. 1 and that at the same instant enters surface No. 2. Designate this current by the symbol I_{12}, *viz.*, put

$$(1.07) \qquad\qquad \frac{dQ}{} = I_1 \qquad\qquad \text{coulombs per second}$$

The quantity dW/dt is the *power input* to the region under consideration attributable to the electropositional forces that act upon the electric charges whose motion constitutes the electric current I_{12} in this region. Designate this power by the symbol P, *i.e.*, put

$$(1.08) \qquad\qquad\qquad\qquad \overline{dt} \qquad\qquad\qquad\qquad \text{watts}$$

An equivalent statement of the relation expressed by Eq. (1.06) is then

$$(1.09) \qquad\qquad\qquad P =$$

The power P defined by this equation is usually referred to as the *electric power input* to the region under consideration.

1.07. Resistivity of Conductors and Electrofrictional Force.— Experiment justifies the assumptions that, wherever there exists a conduction current, the flow of electricity that constitutes this current is always opposed by a force, other than an electropositional force, that depends upon the *density* of the conduction current and the nature of the material at the point under consideration, and that the work done against this force manifests itself as heat energy.

To express this electrofrictional force quantitatively it is necessary to define first the *density* of a conduction current. Note first of all that the *intensity* of such a current through a specified area, *i.e.*, the rate of flow of electricity through a specified area of a conductor, depends upon the orientation of this area. If at any point P an infinitesimal plane surface of area dS is so orientated that the rate of flow of electricity through it at a specified instant of time t is a *maximum*, then the rate of flow, *viz.*, the current dI, through it at this instant *divided by* this area dS

$$(1.10) \qquad\qquad J = \frac{dI}{dS} \quad \text{amperes per square meter}$$

is called the *magnitude* of the *density* of the electric current *at the* point P at time t. The current density at a point is also considered as having a *direction*, *viz.*, the direction of the normal through the surface dS, when orientated as just stated, drawn in the sense of the flow of positive electricity through this surface. The mks unit of current density in the ampere per square meter.

A line drawn in a conductor in such a manner that its direction coincides at each point with the direction of the current density at that point is called a *stream line* of electric current. Through a surface that is tangent at each point to the stream line through this point there is no conduction current.

In terms of the current density J at the points that make up a surface of any shape whatever, the total current through this surface is the *surface integral*

$$(1.11) \qquad\qquad I = \int_S (J \cos \theta) \, dS \qquad\qquad \text{amperes}$$

where dS represents an infinitesimal plane area of this surface at any point, J is the magnitude of the current density at this point, θ is the angle between the direction of this current density J and the normal drawn through the surface at this point in a specified sense, and the integral sign indicates the summation of the products $(J \cos \theta)\ dS$ for all the infinitesimal areas that make up this surface. If I as thus calculated is a positive number then the current is through this surface in the sense of the specified normal; if I as thus calculated is a negative number, then the current is through this surface in the opposite sense.

In terms of the conduction current density as just defined, experiment justifies the assumption that the *electrofrictional force per unit positive charge* at any point in a homogeneous isotropic* conductor has a magnitude proportional to the current density at this point and a direction opposite to the direction of this current density, *i.e.*, opposite to the direction of the stream line at this point. Thus, if J is the current density in a homogeneous isotropic material at a given point P at time t, the electrofrictional force per unit positive charge at this point at this instant has the magnitude

(1.12)† newtons per coulomb

where, for a given material at given temperature and pressure, ρ is a *constant*, independent of the value of J. This constant is called the *electric resistivity* of the given material at the specified temperature and pressure. Except for gases the resistivity is substantially independent of pressure but does depend appreciably upon temperature. The direction of this electrofrictional force is *opposite* to that of the current density J. The mks unit of resistivity is called the ohm-meter.‡ Note from Eq. (1.12)

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* By an isotropic substance is meant a substance that has internally the same properties in all directions.

† Note that, *in this chapter*, the symbol F is used to represent force per unit charge. In subsequent chapters the force per unit charge is identified with the concept of *electric field intensity*, a *vector* for which the conventional symbol is **E**. The symbol F is used in this chapter to avoid confusion with the symbol E, which is used to represent electromotive force (see Sec. 1.08).

‡ The name *ohmmeter* is also used for an instrument (meter) that measures electric resistance in ohms. When it is so used the hyphen is omitted.

that the dimensions of resistivity are

$$(1.13) \qquad \rho \rightarrow \frac{F_r}{J} \rightarrow \frac{\text{newtons/coulomb}}{\text{coulombs/sec (meter)}^2} \rightarrow \frac{L^3 M}{Q^2 T}$$

in which the dimensions are represented by letters as follows:

Length... L
Mass... M
Time... T
Electric charge.................................... Q

The dimensions of electric resistance are $L^2 M Q^{-2} T^{-1}$, and the unit of resistance is the *ohm*, as shown below. This accounts for the name, ohm-meter, of the unit of resistivity.

The work done against the electrofrictional force by whatever agent causes the flow of electricity through a conductor always
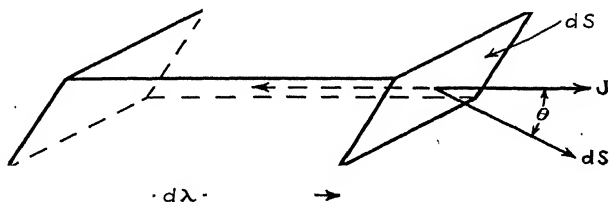
FIG. 1.01.

appears as heat energy. The rate at which heat energy is thus developed per unit volume of the conductor, *i.e.*, the power per unit volume dissipated as heat (attributable to the resistivity of the conductor) at a given point P, is

$$(1.14) \qquad\qquad p_h = \qquad\qquad \text{watts per cubic meter}$$

This relation follows directly from Eqs. (1.11) and (1.12), as will now be shown. Imagine an infinitesimal parallelepiped whose sides are parallel at each point to the stream line of the conduction current through that point. Let J be the conduction current density in this volume, and let θ be the angle between the direction of this current density and the direction of the normal drawn outward from this volume at the end surface at which there is a flow of electricity from this volume. Let $d\lambda$ be the length of each edge of this parallelepiped and let dS be the area of each

end surface. The volume of this parallelepiped is then (Fig. 1.01) *

$$(1.15) \qquad dv = d\lambda \, dS \cos \theta \qquad \text{cubic meter}$$

The quantity of electricity that enters one end surface and leaves the other in time dt is

$$(1.16) \qquad dQ = dI \, dt = (J \, dS \cos \theta) \, dt \qquad \text{coulombs}$$

The electrofrictional force on this charge is, from Eq. (1.12)

$$(1.17) \qquad F_r \, dQ = \rho J \, dQ \qquad \text{newtons}$$

and therefore the work done against this force when this charge moves the distance $d\lambda$ is

$$(1.18) \qquad dW = F_r \, dQ \, d\lambda = \rho J \, dQ \, d\lambda \qquad \text{joules}$$

From Eqs. (1.15) and (1.16) this may be written

$$(1.19) \qquad dW = \rho J^2 \, dv \, dt \qquad \text{joules}$$

The *rate* at which this work is done *per unit volume* is therefore ρJ^2, which proves the relation given by Eq. (1.14).

The total rate at which heat is developed in a given region of space of volume v as the result of the work done against the electrofrictional forces in this region is

$$(1.20) \qquad P_h = \int_{\_} \rho J^2 \, dv \qquad \text{watts}$$

where dv is any infinitesimal volume in this region, J is the magnitude of the conduction current density is this infinitesimal volume, ρ is the electric resistivity of this volume, and the integral sign indicates the summation of the products $\rho J^2 \, dv$ for all the infinitesimal volumes which make up the given volume v.

1.08. Electric Resistance and Resistance Drop.—Consider any volume of a homogeneous isotropic conductor whose lateral walls are tangent at each point to the stream line of the conduction current at that point and whose end surfaces are equipotential. Let the only forces that act upon the free electrons in this volume be the *electropositional* forces, and the electrofrictional forces that oppose the drift of electricity conduction current through this volume. Also let there be no net accumulation of charge within, or on the surface of, this volume. Since by hypothesis the lateral walls of this volume are tangent to the

* The dashed-line arrow in Fig. 1.01 indicates direction of electron drift.

stream lines of the current, the current I_{12} that enters it at one end surface must leave it at the other end surface, and this same current must pass through that part of every intervening equipotential surface which lies within this volume.

Under the conditions just stated the electropositional force at each point of the volume under consideration must be equal in magnitude, but of opposite sign, to the electrofrictional force at this point. Therefore under these conditions the drop of potential V_{12} from the end surface, or *terminal*, at which the current I_{12} enters this volume to the end surface, or *terminal* at which the current leaves this volume has the magnitude

(1.21)   volts

where $d\lambda$ is an infinitesimal length of any stream line that connects a point in the end surface 1 with a point in the end surface 2, and ρ and J are respectively the resistivity and current density at this point, and the integral sign indicates the summation of the products $\rho J \, d\lambda$ for all the infinitesimal lengths that make up this stream line.

Experiment shows that under the conditions just specified this resistance drop V_r, which under these conditions is the *total* difference of potential between the two terminals of the volume under consideration, is proportional to the total conduction current I_{12} that flows through this volume, *viz.*,

(1.22) $V_r =$ volts

where, for the particular size and shape of the volume under consideration, the factor R is a constant, provided the resistivity ρ of this volume is a constant. This factor R is called the *electric resistance* of the given volume between the specified end surfaces, and its mksq unit is called the *ohm*. Note that the ohm has dimensions corresponding to volt/ampere according to Eq. (1.22). Reducing this to a ratio of the four fundamental quantities, it is found that resistance has the dimensions $L^2MQ^{-2}T^{-1}$, as noted in Sec. 1.07. The relation expressed by Eq. (1.22), which may also be written

(1.23)   amperes

is known as *Ohm's law*.

The reader's attention is called particularly to the fact that this relation holds only when there are in the volume under consideration no forces that produce or oppose the drift of electricity through it other than *electropositional* forces (attributable to the magnitudes and positions of electric charges in this volume or in the surrounding regions of space) and the *electrofrictional* forces that manifest themselves in the production of heat energy in this volume. The reader should also note that the resistance of a given volume depends upon the location and shape of the equipotential surfaces that form its end surfaces and upon the location and shape of the stream lines that form its lateral walls.

In general the resistance of a given volume of homogeneous material can be expressed in terms of the resistivity ρ of this material, only when the current density at each point relative to the total current through this volume is known. For example, if J is the current density at a given point when the total current is I ampere, then from Eqs. (1.21) and (1.22),

$$(1.24) \qquad\qquad J \, d\lambda \qquad\qquad \text{volts}$$

where the integration is taken along any arbitrarily chosen stream line from one terminal to the other. In particular, if the volume under consideration is a right cylinder (for example, a specified length of straight insulated wire), and λ is the total length of this cylinder and S is its cross section, then $J = I/S$ and

$$(1.25) \qquad\qquad\qquad\qquad\qquad\qquad \text{ohms}$$

This formula is applicable, to a close approximation, to a wire of length λ and cross section S even though the wire is not straight, provided the linear dimensions of the area S are small compared with the length λ, and also provided that the *only* forces that act upon the drifting electricity are electropositional and electrofrictional forces. The calculation of the resistances of conductors of various shapes is discussed further in Chap. III.

The drop of potential along any stream line of a conduction current, which is given by either of the formulas

$$(1.21) \qquad\qquad r_{12} = \int_1^2 \rho J \, d\lambda \qquad\qquad \text{volts}$$

or

$$(1.22) \qquad\qquad V_{12} = RI_1 \qquad\qquad \text{volts}$$

is called the *resistance drop* along this particular stream line.

When the resistance drops along *all* stream lines of conduction current from one terminal to another of a given volume have the *same* value, this resistance drop may always be written in the form RI_{12}, and the corresponding rate of development of heat energy in this volume due solely to its resistance may be written either in the form

$$(1.20) \qquad\qquad\qquad\qquad ^1 dv \qquad\qquad \_ \quad \text{watts}$$

or

$$(1.26) \qquad\qquad P_h = RI_{12}^2 \qquad\qquad \text{watts}$$

The relation Eq. (1.26) is known as *Joule's law*, and follows directly from Eq. (1.22) and the fact that Eq. (1.20) may be written, in the special case here considered,

$$(1.27) \qquad P_h = \int_1^2 \rho J \, d\lambda \int_S J \, dS = V_{12}I_{12} = (RI_{12})I_{12} \qquad \text{watts}$$

1.09. Electromotive Force—Sources and Receivers.—In general the electric potential drop from a point P_1 on a stream line of the conduction current in a given volume to another point P_2 on the same stream line may be either greater or less than the resistance drop along this line, and there may even be a *rise* of potential instead of a drop of potential in the positive sense of this stream line, which by definition is the positive sense of the conduction current. This arises from the fact that an electric charge at any point of such a stream line may be acted upon by other forces than the electropositional force and the electrofrictional force.

Let F_p be the electropositional force per unit positive charge exerted on an infinitesimal charge dQ at a point P. From Sec. 1.07 the electrofrictional force per unit positive charge exerted on this same charge dQ is $F_r = \rho J$, where ρ is the resistivity at P and J is the current density at this point. Let F_e be the resultant of *all the other forces* per unit positive charge exerted upon this same charge, including the kinetic reaction if this charge is accelerating at an appreciable rate. Let θ_p be the angle

between the direction of F_p and the direction of the stream line at P, and let θ_e be the angle between the direction F_e and the direction of the stream line at P. Then from the principle that action and reaction are always equal and opposite, and that the electrofrictional force at any point P always has a direction *opposite* to that of the stream line of the conduction current at this point,

$$(1.28) \qquad F_p \cos \theta_p + F_e \cos \theta_e = \rho J \quad \text{newtons per coulomb}$$

When a charge dQ is displaced along the stream line in the sense from P_1 to P_2 the amount of work done *by* the electropositional force F_p, *viz.*, the *total drop* of electric potential from P_1 to P_2 is then

$$(1.29) \qquad V_{12} = \int_1^2 (F_p \cos \theta) \, d\lambda \qquad \text{volts}$$

and the amount of work done per unit positive charge by the force F_e is

$$(1.30) \qquad E_{12} = \int_1^2 (F_e \cos \theta_e) \, d\lambda \qquad \text{volts}$$

Under the same conditions the amount of work done per unit positive charge *against* the electrofrictional force $F_r = \rho J$ is the *resistance drop*

$$(1.31) \qquad V_r = \int_1^2 \rho J \, d\lambda \qquad \text{volts}$$

From Eq. (1.28) the three quantities V_{12}, V_r, and E_{12} must bear to each other the relation

$$(1.32) \qquad V_{12} + E_{12} = V_r \qquad \text{volts}$$

The quantity E_{12}, defined by Eq. (1.30) and having from Eq. (1.32) the value

$$(1.33) \qquad E_{12} = V_r - V_{12} \qquad \text{volts}$$

is called *electromotive force.*[*] The word force as here used is actually a misnomer, since the quantity E_{12} is not a force but is *work* per unit charge, just as potential difference is *work* per unit charge. However, the terminology here stated has been used for many years, and it will probably be used for many more.

* Throughout this book *electromotive force* is abbreviated *emf*.

Either of the quantities V_{12} or E_{12} may be positive or negative, dependent upon the directions of the forces F_p and F_e. The potential drop V_{12} is negative if the electropositional force *opposes* the flow of positive electricity from P_1 to P_2, in which case θ_p is greater than 90 degrees, and therefore

$$(1.34) \qquad = -\int_1^2 (F_p \cos \theta) \, d\lambda = -V_{12} \qquad \text{volts}$$

is a positive number. This means that the drop of potential along the specified portion of the stream line from 1 to 2 is in the *opposite* sense to the direction of the conduction current. Similarly, the electromotive force E_{12} is negative if the resultant force F_e which gives rise to it *opposes* the flow of positive electricity from P_1 to P_2, in which case θ_e is greater than 90 degrees, and

$$(1.35) \qquad E_{21} = - \qquad (F_e \cos \theta_e) \, d\lambda = -E_{12} \qquad \text{volts}$$

is a positive number. The relation expressed by Eq. (1.33) can therefore be written in either of the forms

$$(1.36) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{volts}$$
$$(1.37) \qquad\qquad\qquad V_{21} = \qquad - V_r \qquad\qquad \text{volts}$$

in each case V_r is a positive number and gives the *resistance drop* through the given volume in the *positive* sense of the current, *viz.*, in the sense from 1 to 2. If E_{21} is positive, *i.e.*, if the emf in the volume under consideration *opposes* the flow of positive electricity through this volume, then V_{12} is likewise positive, which means that there is an actual *total drop* of potential through this volume in the positive sense of the current in it. On the other hand, if E_{12} is *positive* and greater than V_r then V_{21} is likewise positive. This means that there is an actual *drop* of potential through the given volume in the sense *opposite* to the positive sense of the current through it, or an actual *rise* of potential in the *same* sense as the positive sense of the current through it.

Consider any volume with two end surfaces that have areas dS_1 and dS_2 in two equipotential surfaces whose electric potentials are respectively V_1 and V_2, and whose lateral surfaces are tangent at every point to the stream line of the conduction current at that point, and through which the conduction current is in the sense from dS_1 to dS_2. Let dS be the cross section of this volume at a point, P, and let J be the density of the conduction current at this point.

The conduction current dI_{12} through every infinitesimal cross section dS of this volume at time dt in the sense from dS_1 to dS_2 is then

$$(1.38) \qquad\qquad dI_{12} = J\,dS \qquad\qquad \text{amperes}$$

and is *independent* of the location of this cross section. The electric power input to such a *filament* is then

$$(1.39) \qquad\qquad = V_{12}\,dI_{12} = \quad dI_{12} + V_r \iota \qquad\qquad \text{watts}$$

where E_{21} is the emf in this filament in the sense from dS_2 to dS_1, and V_r is the resistance drop in this filament from dS_1 to dS_2, *viz.*, in the sense of the current dI_{12}.

Consider next a volume made up of contiguous filaments of this kind with all their end surfaces at which the current enters in an equipotential surface S_1 of potential V_1 and all their other end surfaces in an equipotential surface S_2 of potential V_2. The total current through this volume from the *terminal* S_1 to the terminal S_2 is then

$$(1.40) \qquad\qquad I_{12} = \qquad\qquad \text{amperes}$$

where S is the area of that portion of any intervening equipotential surface between these two surfaces S_1 and S_2 that is bounded by the intersection of this equipotential surface with the lateral walls of the given volume. The total electric power input to this volume is then

$$(1.41) \qquad \text{P}_i = \int_S V_{12}J\,dS = \int_S \qquad dS + \int_S V_r J\,dS \qquad \text{watts}$$

where the integrals are surface integrals. Since by hypothesis $V_{12} = V_1 - V_2$ has the *same* value for each of the filaments that make up the given volume, V_{12} is a constant with respect to dS, and therefore

$$(1.42) \qquad \int_S \qquad = V_{12} \int_S J\,dS = \qquad\qquad \text{watts}$$

In like manner, *if the emf E_{21} has the same value for each filament*, then

$$(1.43) \qquad \int_1 J\,dS = E_{21} \int_S J\,dS = \qquad\qquad \text{watts}$$

and

$$(1.44) \qquad V_r J \, dS = V_r \int_S J \, dS = \qquad\qquad \text{watts}$$

The product $V_r I_{12}$ in this case is the rate at which electric power is dissipated as heat energy due to the resistance R of the total volume under consideration, and may be written

$$(1.45) \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{watts}$$

where R has the same value as it would have were there no emfs present. In this case the resistance drop V_r may then be expressed by the simple relation

$$(1.46) \qquad\qquad V_r = RI_1 \qquad\qquad \text{volts}$$

Therefore, when the *emf* E_{21} in each stream line of the conduction current through a given volume from one equipotential surface to another, *viz.*, from one terminal S_1 to a second terminal S_2, *has the same value*, the electric power input to this volume may be written

$$(1.47) \qquad P_i = V_{12}I_{12} = E_{21}I_{12} + RI_{12} \qquad\qquad \text{watts}$$

and the voltage drop from S_1 to S_2 can be written

$$(1.48) \qquad\qquad V_{12} = E_{21} + RI_{12} \qquad\qquad \text{volts}$$

which is identical with Eq. (1.36). Under the same conditions, if the electric power input is *negative, i.e.,* if there is actually an electric power output $P_o = -P_i$, then this power output may be written

$$(1.49) \qquad\qquad P_o = E_{12}\,I_{12} - RI_{12}^2 \qquad\qquad \text{watts}$$

and the voltage drop from S_2 to S_1 may be written

$$(1.50) \qquad\qquad V_{21} = E_{12} - RI_{12} \qquad\qquad \text{volts}$$

which is identical with Eq. (1.37). The power RI_{12}^2 in both Eqs. (1.47) and (1.49) is the rate at which electric energy is converted into heat energy within the volume under consideration, *owing to its resistance.* In Eq. (1.47), the power $E_{21}I_{12}$ is the rate at which *electric energy is converted into some other form of energy,* such as chemical, mechanical, magnetic, electrostatic, etc. The total power input in this case is the sum of the electric power so con-

verted and the electric power dissipated as heat with the given volume.   In Eq. (1.49) the power $E_{12}I_{12}$ is the rate at which some other form of energy is *converted into electric energy* within the given volume.   In the latter case the power output is this so-called *generated* electric power minus the power dissipated as heat within this volume.

A volume of the type just described will hereafter be referred to as a *single-branch source* of electric energy when it is so employed that electric energy is generated within it, and a *single-branch receiver* of electric energy when it is so employed that electric energy is converted within it into some other form of energy (such as heat energy, mechanical energy, or chemical energy).

1.10. Simple Electric Circuit.—By an *electric network* is meant two or more conductors whose lateral walls are insulated (so that no conduction current passes through them) and whose end surfaces are so connected that these conductors form one or more *closed meshes*.   When the end surfaces of each such conductor are equipotential surfaces and the emf, if any, in each stream line through this conductor has the same value, this conductor is called a *branch* of the network.   The end surfaces of each branch are called the *terminals* of this branch.   Since the linear dimensions of the cross section of a branch are usually small compared with the length of the steam lines through it, the junction of two such end surfaces is usually called a *junction point*.

An electric cell or battery, such as a dry cell or a wet cell, may as a rule be considered as a single branch.   A single wire, straight, curved, or coiled, may also as a rule be considered as a single branch, unless the current in it is varying extremely rapidly with respect to time.   The armature winding of a simple electric generator or motor may also be considered as a single branch and its field winding as a second single branch.

A parallel-plate condenser,* with each plate connected conductively to a single terminal in an electric network, is also considered as a *branch,* although all the conduction current that enters one plate and leaves the other does not actually pass *through* the dielectric that separates them.   Such a device is characterized by the fact that the conduction current that enters

* This can be thought of for present purposes as two plates of a good conductor separated by a sheet of dielectric whose thickness is extremely small compared with the linear dimensions of the conducting plates.

one plate (called the *positive* plate) is always equal to the conduction current that leaves the other plate (called the *negative* plate), but within the dielectric there is either no conduction current, or only a relatively small conduction current. If there is no conduction current in the dielectric, *i.e.*, if the dielectric is a *perfect* insulator, then the conduction current I that enters one plate at any instant is equal to the time-rate of increase of the charge Q on the positive plate, *viz.*,

$$(1.51) \qquad\qquad \frac{dQ}{dt} \qquad\qquad \text{amperes}$$

This current is usually referred to as the *charging current* of the condenser.

If the dielectric is a partial conductor, the voltage drop V_c from the positive plate to the negative plate is always accompanied by a conduction current I_1 through this dielectric, which conduction current may be expressed as the product of V_c by a factor G, *viz.*,

$$(1.52) \qquad\qquad I_1 = GV_c \qquad\qquad \text{amperes}$$

This factor G is called the *leakage conductance* of the condenser, and the current GV_c is called the *leakage current*. For a non-varying potential difference V_c this factor G is the *reciprocal* of the electric resistance of the dielectric between the two plates; when V_c varies with time this factor in general has a value that increases with the rapidity of the variation in V_c. In any case, the total conduction current that enters the positive plate of the condenser and leaves the negative plate, at any instant of time, is

$$(1.53) \qquad\qquad\qquad\qquad\qquad\qquad \text{amperes}$$

where Q is the charge on the positive plate at this instant and V_c is the voltage drop from the positive to the negative plate. In a *perfect* condenser G is zero, and Eq. (1.53) reduces to Eq. (1.51).

When two branches are connected end to end so that they form a closed loop, and the conduction current at any instant that enters each junction of terminals through one branch is the same as the current that leaves this junction through the other branch, these two branches are said to form a *single-mesh network* or a *simple electric circuit*. A special case of a simple electric circuit is shown in Fig. 1.02, which represents diagrammatically a branch

on the left that contains an emf E_1 and that has a total resistance R_1, and on the right a branch that contains an emf E_2 and a resistance R_2. These two branches have the common terminals A and B. As indicated by the arrows on the diagram the emf E_1 in the left-hand branch has a positive sense *through this branch* from B to A, and the emf E_2 in the right-hand branch has a positive sense *through this right-hand* branch also from B to A. Con-
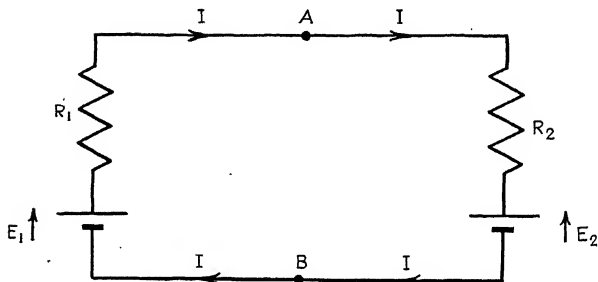
FIG. 1.02.

sidered from the point of view of the *closed loop* formed by these two branches, the emf E_1 is in the *clockwise direction* around this loop and the emf E_2 in the *counterclockwise direction*. From this circulatory point of view these two emfs may be considered as *opposing*, or "bucking," each other. If the current is actually in the direction indicated by the arrows at the terminals, then E_1 is said to be the *driving* emf, and E_2 is said to be a *back* emf.

In the simple electric circuit depicted in Fig. 1.02 the voltage drop from the terminal A to the terminal B may be written either

$$(1.54) \qquad\qquad V_{AB} = E_1 - R_1 I \qquad\qquad \text{volts}$$

or

$$(1.55) \qquad\qquad V_{AB} = \qquad\qquad\qquad \text{volts}$$

These relations follow directly from Eqs. (1.48) and (1.50), noting that I in Eq. (1.54) is the current *through* the left-hand branch from B to A, and that I in Eq. (1.55) is the current *through* the right hand branch from A to B. From the circulatory point of view the current I in both expressions is the current around the closed loop in the direction of the emf E_1, *viz.*, in the clockwise direction.

Since the drop of potential V_{AB} is independent of the path from A to B, the right-hand members of the two expressions Eqs. (1.54) and (1.55) are equal,

$$(1.56) \qquad E_1 - R_1 I = E_2 + R_2 I \qquad \text{volts}$$

Therefore the current I must have the value

$$(1.57) \qquad \frac{E_1 - E_2}{R_1 + R_2} \qquad \text{amperes}$$

If E_1 is greater than E_2, then I is a positive number, which means that I is actually in the sense indicated in Fig. 1.02 and therefore the left-hand branch in the figure is a *source*, and the right-hand branch is a *receiver*.   If E_1 is less than E_2, then the number given by Eq. (1.57) is *negative*, which means that the current is actually in the sense opposite to the arrow alongside I in the diagram, and therefore the right-hand branch is the source and the left-hand branch the receiver.   In this case the current I' in the counterclockwise direction is the positive number

$$(1.58) \qquad I' = -I = \qquad \qquad + R_2 \qquad \text{amperes}$$

If the left-hand branch in Fig. 1.02 is the source, its electric-power output is

$$(1.59) \qquad P_{1o} = V_{AB} I = E_1 I - R_1 I^2 \qquad \text{watts}$$

This is also the electric-power input to the receiver formed by the right-hand branch, which may also be written

$$(1.60) \qquad P_{2i} = V_{AB} I = E_2 I + R_2 I^2 = P_{1o} \qquad \text{watts}$$

If the right-hand branch is the source, its electric-power output is

$$(1.61) \qquad P_{2o} = V_{AB} I' = E_2 I' - R_2 (I')^2 \qquad \text{watts}$$

This is also the electric-power input to the receiver formed by the left-hand branch, which may also be written

$$(1.62) \qquad P_{1i} = \qquad \qquad \text{watts}$$

1.11. Branches in Series.—Designate by AB a branch between two terminals A and B, and by BC a branch that is connected to AB at the terminal B, as shown in Fig. 1.03.   Designate by R_1 and R_2 the resistance of these two branches and by E_1 and E_2

the emfs in these two branches in the sense from A to C.   Designate by I the current that enters the terminal A and leaves the terminal B.   If there is no change in the net electric charge at the common terminal B, then the same current I leaves the branch AB at the terminal B and enters the branch BC at the same terminal, and the same current I leaves the branch BC at the terminal C.   The two branches AB and BC are then said to be in *series*.
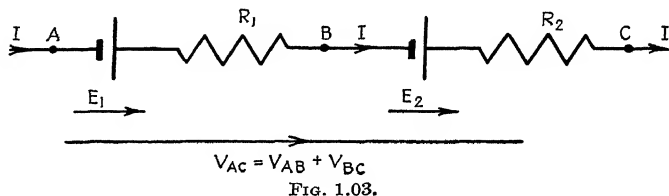
$$V_{AC} = V_{AB} + V_{BC}$$
Fig. 1.03.

Since the voltage drop from any point A to another point C is independent of the path from A to C, this voltage drop, no matter what other conducting paths may exist from A to C, is

$$(1.63) \qquad \begin{aligned} V_{AC} &= V_{AB} + V_{BC} \\ &= -E_1 + R_1 I - E_2 + R_2 I \end{aligned} \qquad \text{volts}$$

or

$$(1.64) \qquad V_{AC} = \qquad + E_2) + (R_1 + R_2)I \qquad \text{volts}$$

The voltage drop V_{AC} given by Eq. (1.64) is identical with that which would exist were the two branches AB and BC in series replaced by a *single branch AC* having the resistance

$$(1.65) \qquad R = R_1 + R_2 \qquad \text{ohms}$$

and containing an emf in the sense from A to C equal to

$$(1.66) \qquad E = E_1 + E_2 \qquad \text{volts}$$

The logical extension of this analysis to any number of branches in series leads to the general rule:

Any number of branches (other than electric condensers) in series is equivalent to a single branch whose resistance is the sum of the resistances of the individual branches and which contains a single emf equal to the algebraic sum of the emfs in the individual branches.

The equivalence here stated is valid not only with respect to the relation between the current I and total voltage drop V_s from one end of the series to the other, but also with respect to the total electric energy generated (or absorbed in other forms than heat energy due to the several resistances) and to the total energy dissipated in this series as heat energy; *i.e.*, if I is the current in this series and

$$(1.67) \qquad\qquad E = \qquad\qquad\qquad\qquad \text{volts}$$

is the emf of this equivalent single branch in the sense of this current and

$$(1.68) \qquad\qquad\qquad\qquad + \qquad\qquad\qquad \text{ohms}$$

is the resistance of this equivalent single branch, and V is the voltage drop through the series from one end to the other in the sense of the current I, then RI^2 is the rate at which electric energy is dissipated as heat energy in all the elements of this series because of their resistances and EI is the rate at which electric energy is converted in these elements into any form of energy other than the heat energy due to their resistances.

When one of the branches that make up a series is an electric condenser, and there is no time variation of the charge on its plates, this branch is nothing more than a branch of resistance $1/G$, where G is the conductance of the dielectric of this condenser. To a non-varying electric current an electric condenser is therefore either a substantially open circuit (G negligibly small) or a branch of very high resistance. The effect of an electric condenser in series with one or more branches, when the current is varying with time, will be considered in detail in a later section.

1.12. Branches in Parallel.—Designate by $(AB)_1$ and $(AB)_2$ two branches whose A terminals are connected and whose B terminals are connected in the manner shown in Fig. 1.04. Designate by V_{AB} the voltage drop from terminal A to terminal B. If there is no change in the net electric charge at either terminal, then the total current I that enters these two branches at terminal A must be equal to the current that leaves them at terminal B. The two branches $(AB)_1$ and $(AB)_2$ are then said to be *in parallel.*

Designate by R_1 and R_2 the resistances of these two branches and by E_1 and E_2 the emfs in them in the sense from A to B, and let there be no electric condenser in either branch.

nate by I_1 and I_2 the conduction currents in these two branches in the sense from A to B. Since the voltage drop from A to B is independent of the path from A to B, this voltage drop is

$$V_{AB} = -E_1 + R_1 I_1$$

(1.69)
$$= -E_2 + R_2 I_2 \qquad \text{volts}$$

and therefore

(1.70)
$$\frac{V_{AB} + E_1}{R_1} \qquad \text{amperes}$$

(1.71)
$$\frac{V_{AB} + E_2}{R_2} \qquad \text{amperes}$$

Since by hypothesis there is no change in the net electric charge at either of the terminals A and B, the sum of these two currents I_1 and I_2 must be equal to the total current I. Therefore

(1.72)
$$I = \frac{V_{AB} + E_1}{R_1} + \frac{V_{AB} + E_2}{R_2} \qquad \text{amperes}$$
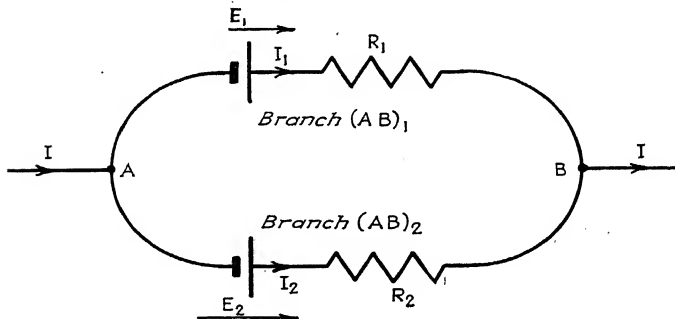
Fig. 1.04.

In the special case when the two emfs E_1 and E_2 have the *same value* E and are in the *same sense*, this last relation can be written

(1.73)
$$I = \left(\frac{1}{R_1} + \frac{1}{R_2} \qquad \cdot E \right) \qquad \text{amperes}$$

This relation is identical with that which would hold were the two parallel branches replaced by a single branch having a resistance R such that

(1.74)
$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \qquad \text{(ohms)}^{-1} \text{ or mhos}*$$

* The unit of conductance, the reciprocal ohm, is called a *mho*.

and containing the same emf E as each of the actual branches. The logical extension of this analysis to any number of branches in parallel leads to the rule:

Any number of branches (other than electric condensers) in parallel, between two terminals when the emfs (if any) in all branches have the same value and the same sense relative to these terminals, is equivalent to a single branch whose resistance is such that its reciprocal is the sum of the reciprocals of the resistances of the individual branches, and which contains the same emf as each of these branches.

As in the case of branches in series, the equivalence just stated is valid not only with respect to the relation between the total current I and the voltage drop V through each branch, but also with respect to the total electric energy generated (or absorbed in other forms than heat energy due to the several resistances), and also with respect to the total energy dissipated as heat energy in the several branches due to their resistances.

As in the case of branches in series, when one or more of the several parallel branches is an electric condenser and there is no time variation of the charge on its plates, this branch is nothing more than a branch of resistance $1/G$, where G is the conductance of the dielectric of this condenser. The effect of an electric condenser in parallel with one or more branches, when the current is varying with time, will be considered in detail in a later section.

The reader's attention is called particularly to the fact that when the emfs in several parallel branches are not equal, these parallel branches are *not equivalent internally* to a single branch. However, relative to the rest of the network to which such a parallel group is connected, it may be replaced by a single branch, but in general the product of the resistance of this *externally* equivalent branch by the square of the total current I through it does not give the power dissipated as heat in the actual parallel branches. The single branch that is externally equivalent to a two-terminal network will be discussed in detail in a later section.

Equation (1.74) for the equivalent resistance R of *two* branches in parallel can also be expressed in the form

$$(1.75) \qquad R = \frac{R_1 R_2}{R_1 + R_2} \qquad \text{ohms}$$

1.13. Network Parameters and Generated Electromotive Forces.—It has already been pointed out that when a conduction current enters one plate of a condenser and passes out of the other plate, without actually passing through the dielectric of the condenser, a positive electric charge accumulates on the first plate and a numerically equal negative charge accumulates on the second. In the electron theory this is accounted for on the assumption that there occurs a drift of electrons from the first plate to the second through whatever external source of emf supplies the electricity necessary to produce this drift.

It is an experimental fact that for a condenser of fixed dimensions, temperature, pressure, etc., the charge that is on one plate P_1 at any instant of time t is proportional to, or at least approximately proportional to, the voltage drop V_c from this plate to the other plate P_2, *i.e.*,

$$(1.76) \qquad\qquad Q = CV_c \qquad\qquad \text{coulombs}$$

where C is a positive quantity. The conduction current that enters this plate P_1 from the outside source is then, for C a constant,

$$(1.77) \qquad\qquad I_c = C\frac{dV_c}{dt} \qquad\qquad \text{amperes}$$

As already noted, this current is called the *charging current* of the condenser. The factor C is called the *capacitance* of the condenser, and its mksq unit is called the *farad*. Note that *a charging current exists only when the condenser voltage is varying with time*. The value of the capacitance of a condenser depends upon the dimensions and geometrical shape of the dielectric that separates the conducting plates, and also upon the dimensions of the surfaces of these plates in contact with the dielectric. In the case of certain simple geometric shapes relatively simple formulas have been developed for the computation of capacitance, based on the concepts of the electric-field theory discussed in detail in a later chapter. Methods have also been developed for the measurement of capacitance; see any comprehensive electrical engineering handbook.

It is also an experimental fact that when the conduction current in a conductor, such as an insulated wire (straight, curved, or coiled), is varying with time an *emf* exists within this conductor

in the sense *opposite* to the sense of this current when the latter is *increasing*, and in the same sense as this current when the latter is decreasing.   Also, when the conduction current in a neighboring conductor, or in some other part of this same conductor, is changing with time, an additional emf is set up in the first conductor.   The emfs that are attributable to the time variation of the currents in the various branches of a network, as distinct from emfs that may arise (*a*) from the *motion* of these conductors, or (*b*) from the chemical or physical dissimilarity at the contact between two conductors, are called *induced* emfs.   An emf of either of the types (*a*) or (*b*) will hereafter be referred to as a *generated* emf.

In general, if I_1 is the current at any instant in a given branch of a network, and I_2, I_3, . . . , are the currents in the other branches of this network, the induced emf that *opposes* the current I_1 at any instant in this given branch may be expressed by the relation

(1.78)   volts

where L_1, M_{12}, M_{13}, . . . , are positive quantities.   The first of these factors, *viz.*, L_1, is called the *coefficient of self-induction*, or simply the *self-inductance*, of the given branch, and the others are called the *coefficients of mutual induction*, or *mutual inductance*, between this branch and the other branches.   The appropriate sign to use before each of the mutual-inductance terms depends upon the choice of the positive senses of the currents I_2, I_3, etc., relative to the positive sense of I_1, and also upon the orientation of the branches 2, 3, etc., relative to branch 1.   The mks unit of inductance (both self and mutual) is called the *henry*.

The value of the self-inductance L_1 depends upon the dimensions and geometrical shape of the branch and upon the nature of the magnetic material, if any, both within and outside this branch.   The values of the mutual-inductance coefficients also depend upon these factors, and in addition upon the dimensions, geometric shape, and orientation of these other branches relative to the given branch.   When there are no ferromagnetic materials present and when there is no relative motion of the various parts of the network under consideration, the induction coefficients for any given branch are constants, or substantially constants.

In the case of conductors of simple geometrical shapes, formulas have been developed for the computation of inductance, based upon the concepts of the magnetic-field theory discussed in detail in a later chapter. Methods have also been developed for the measurement of inductance; see any comprehensive electrical engineering handbook.

It follows from the material thus far presented that in general every branch of an electrical network is characterized by one or more of the five quantities:

R = resistance, ohms
G = leakage conductance if branch is a condenser, mhos
L = self-inductance, henrys
M = mutual inductance, henrys
C = capacitance if branch is a condenser, farads

These five quantities for the various branches that make up a network are called the *parameters* of the network. When their values are independent of time and of the conduction currents and condenser voltages in the network, the network is said to have *constant parameters*.

1.14. Kirchhoff's Laws for Electric Networks.—In the discussion of branches in series and parallel use was made of two fundamental principles, *viz.*,

1. The algebraic sum of the electric currents that enter any junction point in a network, when there is no change in the net electric charge at this point, is always zero.

2. The algebraic sum of the voltage drops in the several branches that form a *closed* mesh in a network, in a specified circulatory sense around this mesh, is always zero.

These two principles are perfectly general and are applicable irrespective of whether the network parameters are constants or variables, and also whether or not the currents and potential differences are constants with respect to time or are functions of time, and also irrespective of the interconnections of the several branches that make up an electric network. They are known respectively as *Kirchhoff's first and second laws* for electric networks. They form the basis for all electric-network calculations.

In accordance with Kirchhoff's first law the current I_{12} in any branch of a network that is common to two meshes 1 and 2 of a network may be considered as the *algebraic sum* of two so-called *mesh currents* I_1 and I_2 that have respectively the same value and

the same circulatory sense in each branch of the particular mesh to which it refers. For example, consider the three-mesh network shown diagrammatically in Fig. 1.05, and designate by I_1, I_2, and I_3 the mesh currents in the three small triangular meshes that form the network bounded by the large triangle, and
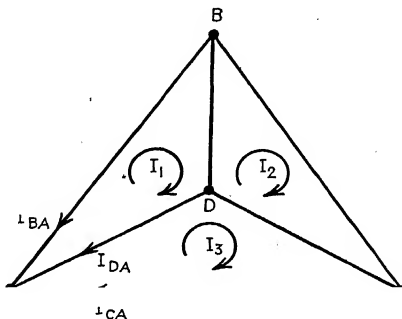

Fig. 1.05.

choose as the positive sense of each of these mesh currents the clockwise direction around the mesh in which it is assumed to exist. The currents in the three branches that have the common terminal A, in the sense *toward* this terminal, are then

$$(1.79) \hspace{3cm} I_{DA} = I_1 - \hspace{3cm} \text{amperes}$$

The sum of these three currents I_{BA}, I_{DA}, and I_{CA} is identically zero. Again, the currents in the three branches which have terminal D as a common terminal, in the sense toward that terminal, are

$$(1.80) \hspace{2cm} \begin{aligned} I_{AD} &= I_3 - I_1 \\ I_{BD} &= I_1 - I_2 \\ I_{CD} &= I_2 - I_3 \end{aligned} \hspace{2cm} \text{amperes}$$

The sum of these three currents I_{AD}, I_{BD}, and I_{CD} is also identically zero.

   This concept of mesh currents greatly simplifies many mathematical calculations. The assumption of the existence of these hypothetical mesh currents guarantees that the branch currents

will satisfy Kirchhoff's first law. It also leads to a relatively simple notation for the current in any branch, *viz.*, that the current in a branch that is common to two meshes can be designated by a double numerical subscript indicating two of the meshes to which the branch in question is common, and the order in which these subscripts are written may be so chosen that the first (left-hand) subscript is that of the mesh current, which in this branch has the same positive sense as that of the branch current so designated. For example, in Fig. 1.05 the branch current in the branch from A to D, in the sense from A to D, may be designated I_{31}, since the positive sense of the mesh current I_3 is so chosen that in this branch this positive sense is from A to D. Similarly, I_{13} designates the current in this same branch in the sense from D to A, so that

$$(1.81) \qquad\qquad I_{13} = -I_{31} \qquad\qquad \text{amperes}$$

This convention of a double-subscript notation for branch currents may also be extended to "outside" branches by using 0 as one of the two subscripts. For example, the current in the branch BC in Fig. 1.05 in the sense from B to C is designated I_{2o} and the current in this branch in the opposite sense, *viz.*, from C to B is designated I_{02}. In Fig. 1.05 one then has

$$(1.82) \qquad\qquad \begin{aligned} I_{BC} &= I_{20} \\ I_{CB} &= I_{02} \end{aligned} \qquad\qquad \text{amperes}$$

The convention just stated in regard to branch and mesh currents will be used throughout this book. Also, unless definitely stated otherwise, each mesh current in a plane diagram will always be taken in the clockwise sense around the mesh to which it refers.

The mathematical formulation of Kirchhoff's second law may be effected in a number of ways. In the general case when the currents and condenser voltages are varying with time, the total voltage drop around a closed mesh in a network, which by Kirchhoff's second law is *always* zero, is conveniently considered as the algebraic sum

$$(1.83) \qquad\qquad U \qquad\qquad \text{volts}$$

where E is the resultant *generated emf* in this mesh in the clockwise sense; V_r is the resultant *resistance drop* around this mesh

in this same sense in all those branches in which there are no condensers; E_i is the resultant *induced* emf in this mesh in the *counterclockwise* sense; and U is the resultant of the voltage drops, in the clockwise sense around this mesh, through the condensers (if any) in its several branches. Since the sum Eq. (1.83) is always *zero* for a *closed* mesh, it follows that

$$(1.84) \qquad\qquad\qquad U \qquad\qquad\qquad \text{volts}$$

for each closed mesh in an electric network.

The induced emf E_i in the sense *opposite* to a specified circulatory sense is commonly referred to as the *inductance drop* in this specified circulatory sense. The voltage drop U through the condensers (if any) in a specified circulatory sense is conveniently referred to as the *capacitance drop* in this sense. Hence Eq. (1.84) is the statement, in mathematical form, that around every closed mesh in an electric network the resultant generated emf in a specified circulatory sense is equal to the algebraic sum of (1) the resultant resistance drop through all the branches of this mesh that are not condensers, (2) the resultant inductance drop through all these branches, and (3) the resultant capacitance drop through all the branches that are condensers, all these drops being taken in the specified circulatory sense.

In the special case *when the currents in all parts of the network are constant, and there are no condensers present*, the inductance drop E_i and the capacitance drop U are both zero for each mesh, and Eq. (1.84) becomes

$$(1.85) \qquad\qquad\qquad E = V_r \qquad\qquad\qquad \text{volts}$$

The resistance drop around any closed mesh, say mesh 1, can always be expressed in terms of the mesh currents in this mesh and in the adjacent meshes, and the resistances of the branches that are common to this mesh and these adjacent meshes. Referring to Fig. 1.06, designate by R_{12}, R_{13}, . . . , the resistances of the several branches of mesh 1. Then the resultant resistance drop around this mesh in the clockwise sense is

$$(1.86) \qquad V_{r1} = R_{12}(I_1 - I_2) + R_{13}(I_1 - I_3) + \qquad \text{volts}$$

Put

$$(1.87) \qquad\qquad R_{11} = R_{12} + R_{13} + \cdots \qquad\qquad \text{ohms}$$

Then Eq. (1.86) may be written

$$(1.88) \qquad V_{r1} = R_{11}I_1 - R_{12}I_2 - R_{13}I_3 - \cdots \qquad \text{volts}$$

In a similar manner if the generated emfs in the several branches of mesh 1 in the clockwise sense around this mesh are designated $E_{12}, E_{13}, \ldots$, then the resultant generated emf in this mesh is

$$(1.89) \qquad E_1 = E_{12} + E_{13} + \cdots \qquad \text{volts}$$

Therefore, for the special case of constant currents and no condensers, the mathematical formulation of Kirchhoff's second law is that for each mesh there exists an equation of the form

$$(1.90) \qquad {}_1 + E_{13} + \cdots = R_{11}I_1 - \qquad - R_{13}I_3 - \qquad \text{volts}$$

This is nothing more than the statement that, under the conditions specified, the resultant generated emf in every closed mesh
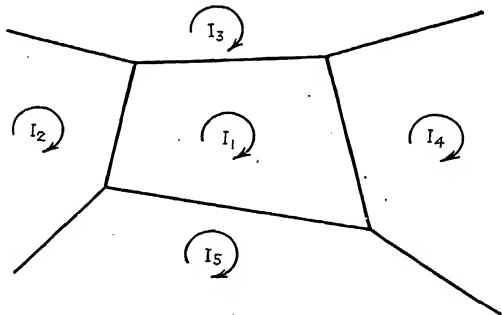

Fig. 1.06.

of a network is equal to the algebraic sum of the resistance drops around this network, the emfs and currents being taken in the same circulatory sense around the given mesh.

The resistance R_{11} defined by Eq. (1.87) is sometimes referred to as the total *mesh* resistance, as contrasted with the *branch* resistances $R_{12}, R_{13}, \ldots$. If a particular mesh k, unlike mesh 1 of the circuit shown in Fig. 1.06, has a branch in which only one mesh current flows, its resistance can be designated R_{k0}, and this term R_{k0} is to be included in the sum $(R_{k1} + R_{k2} + \cdots)$, which is equal to R_{kk}.

A group of simultaneous equations of the form Eq. (1.90) is the basis for all *constant-current calculations*. Constant currents,

i.e., currents which do not vary appreciably with time, are usually referred to as *direct currents*, and calculations relative to their distribution in a network are called *direct-current calculations.* Such calculations will be considered in detail in the next chapter. In subsequent chapters methods will be developed for calculating currents that vary with time in any specified manner.   In the discussion of varying currents it is necessary to take into account not only the resistance R of each branch of the network but also the other four parameters L, M, G, and C of each branch.   Equation (1.84) is always applicable, irrespective of how the currents vary with time, but the expressions for E_i and U in terms of these circuit parameters and the currents are too involved to be considered in this introductory chapter.

1.15. Illustrative Problem.—An alternating-current motor is direct-connected to a direct-current generator; the generator is connected by means of two wires to three storage batteries connected in parallel, in order to charge them.   The mechanical power output of the a-c motor is 0.5 hp.; 5 per cent of this power supplies friction-and-windage losses of the generator. The internal resistance of the generator is 0.02 ohm; the internal resistances of the three batteries A, B, C are 0.09 ohm, 0.08 ohm, 0.07 ohm; their emfs are 5.8 volts.   Two wires, each 20 ft. long, connect the generator to the batteries; these wires are No. 4 A.W.G. copper wires.   An ammeter in series with the generator has a shunt consisting of a piece of constantan* having a cross section of 0.1 sq. in. and a length of 2.5 in.   The resistivity of constantan is approximately 0.49 microhm-meter.   Calculate: (*a*) the current through the generator; (*b*) the generated emf and the terminal voltage of the generator; (*c*) the current through each of the batteries; the rate of dissipation of heat (*d*) in the generator, (*e*) in the wires from the generator to the batteries, (*f*) in the ammeter shunt, and (*g*) in each of the batteries; (*h*) the energy in kilowatt-hours supplied to each battery in 1 hr.; (*i*) the quantity of electricity in ampere-hours passing through each battery in 1 hr.; (*j*) the average drift velocity of the free electrons in the wires from the generator to the batteries; (*k*) the average drift velocity of the free electrons in the ammeter shunt.   Calculate (*j*) and (*k*) on the assumptions that there are 8.5×10^{28} atoms per cu. m. of copper and 1 free electron per atom; and 8.7×10^{28} atoms per cu. m. of constantan and 1 free electron per atom.

Solution.—First calculate the resistance of 40 ft. of No. 4 wire.   From a wire table obtain the resistance of No. 4 copper wire — 0.2485 ohm per 1,000 ft.   The line resistance for both wires (40 ft.) is therefore 0.00994 ohm. Next calculate the resistance of the ammeter shunt from Eq. (1.25),

---

* Constantan is an alloy of copper and nickel whose resistivity changes very little (about 18 parts in 1 million per degree centigrade) with temperature.

$$R_s = \frac{\rho\lambda}{S} = \frac{(0.49 \times 10^{-6})(6.23 \times 10^{-2})}{6.45 \times 10^{-5}} = 4.7 \times 10^{-4} \qquad \text{ohm},$$

changing the given data for λ and S to meters and square meters.

If 5 per cent of the power from the motor is wasted as friction, the mechanical power transformed by the generator is

$$(0.95)(0.5)(746) = 354.35 \qquad \text{watts}$$

$$(\%)(\text{horsepower}) \left(\frac{\text{watts}}{\text{-----------}} \right) \to (\text{watts})$$

Therefore the power input to the electric circuit is 354.35 watts; and this power is equal to the emf of the generator E multiplied by the current I through it. Since the emfs of the batteries are equal (5.8 volts) they can be represented in the circuit by an emf of 5.8 volts in series with a resistance R calculated from Eq. (1.74) in terms of the resistances (0.07 ohm, 0.08 ohm, 0.09 ohm) of the batteries.

$$\frac{1}{R} \quad \frac{1}{0.07} + \frac{1}{0.08} + \frac{1}{0.09} \qquad \text{or} \qquad R = 0.0264 \qquad \text{ohm}$$

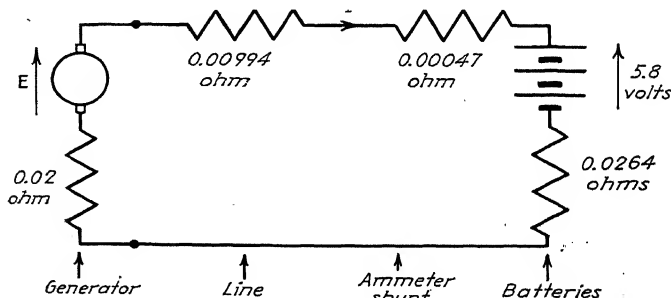Therefore the electric circuit can be represented by the diagram shown in Fig. 1.07.

Fig. 1.07.

The current I can be calculated from the relation

$$EI = 354.35$$
$$EI = (0.02 + 0.00994 + 0.00047 + 0.0264)I^2 + 5.8I$$

(Electric power input) = (Power dissipated as heat) + (Power converted into chemical power)

The solutions a to i can be now calculated.

  a. From the two relations above, eliminating E;

b. The generated emf is

$$\frac{354.35}{3.^\wedge} = 8.24 \qquad \text{volts}$$

The resistance drop in the generator is $(0.02)(43.0) = 0.86$, so that the generator terminal voltage is

$$E - R_G I = 7.38 \qquad \text{volts}$$

c. The terminal voltages of the batteries are equal; call this voltage V, so that

$$V \approx \quad 7.38 \quad - (0.00994)I - (0.00047)I$$

(Battery   (Generator   (Drop in   (Drop in
terminal $\approx$ terminal $-$ line) $-$ ammeter)
voltage)   voltage)

$$V = 7.38 - 0.43 - 0.02 \approx 6.93 \qquad \text{volts}$$

Therefore the currents in the batteries are $\dfrac{V - 5.8}{R_B}$ where R_B is the resistance of the particular battery.

Current in battery $A \approx 12.56$ amp.
Current in battery $B \approx 14.13$ amp.
Current in battery $C \approx 16.14$ amp.

d. Heat dissipated in generator

$$P_G = (0.02)(43)^2 \qquad \approx 37.0 \qquad \text{watts}$$

e. Heat dissipated in No. 4 wires

$$P_L = (0.00994)(43)^2 \approx 18.5 \qquad \text{watts}$$

f. Heat dissipated in the ammeter

$$P_M = (0.00047)(43)^2 \approx \ \ 0.87 \qquad \text{watts}$$

g. Heat dissipated in battery A

$$P_A = (0.09)(12.56)^2 \approx 14.2 \qquad \text{watts}$$

Heat dissipated in battery B

$$P_B = (0.08)(14.13)^2 \approx 16.0 \qquad \text{watts}$$

h. Energy to battery A

$$W_A = \qquad \qquad \text{٠ 0.087} \qquad \text{kw.-hr. per hr.}$$

Energy to battery B

$$\frac{14.}{1000} \qquad \qquad \text{kw.-hr. per hr.}$$

Energy to battery C

$$W_C \cdot \ \frac{}{1000} \ \doteq 0.112 \qquad \text{kw.-hr. per hr.}$$

i. Charge through battery A

$$Q_A = 12.56 \qquad \text{amp.-hr. per hr.}$$

Charge through battery B

$$Q_B = 14.13 \qquad \text{amp.-hr. per hr.}$$

Charge through battery C

$$Q_C = 16.14 \qquad \text{amp.-hr. per hr.}$$

j. To calculate drift velocities note that the current density J is equal to the volume-charge density* ρ multiplied by the drift velocity v.

$$J = \rho v$$
$$\left(\frac{\text{coulombs}}{\text{second (meter)}^2}\right) \rightarrow \left(\frac{\text{coulombs}}{(\text{meter})^3}\right)\left(\frac{\text{meters}}{\text{second}}\right)$$

The cross-sectional area of No. 4 wire is 2.1×10^{-5} square meter so that the current density is:

$$J = \frac{43}{2.1 \times 10^{-5}} = 2.05 \times 10^6 \qquad \text{amp. per m.}^2$$

The volume-charge density of free electrons is

$$\rho = (8.5 \times 10^{28})(1.59 \times 10^{-19}) = 13.5 \times 10^9 \qquad \text{coulombs per cu. m.}$$
$$\left(\frac{\text{coulombs}}{(\text{meter})^3}\right) \quad \cdot \left(\frac{\text{electrons}}{(\text{meter})^3}\right)\left(\frac{\text{coulombs}}{\text{electrons}}\right)$$

Therefore the drift velocity v is

$$\vdots \times 10^{-4} \qquad \text{m. per sec; or } 0.152 \qquad \text{mm. per sec.}$$

k. The calculation follows the pattern for (*j*) above. The drift velocity of the free electrons in the ammeter shunt is approximately 0.048 mm. per sec.

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* Note that, in spite of the confusion which arises from the practice, the symbol ρ is commonly used for both resistivity and for volume-charge density.

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SAMPLE CONVERSION TABLES

Electric Charge or Electric Current

to obtain ↓ by→ Multiply number of →	amperes or coulombs (mks)	abamperes or abcoulombs (cgs)	statamperes or stat-coulombs (cgs)
amperes or coulombs (mks)	1	10	3.335×10^{-10}
abamperes or abcoulombs (cgs)	0.1	1	3.335×10^{-11}
statamperes or statcoulombs (cgs)	2.998×10^{9}	2.998×10^{10}	1

Electromotive Force or Potential Difference

to obtain ↓ by→ Multiply number of →	volts (mks)	abvolts (cgs)	statvolts (cgs)
volts (mks)	1	10^{-8}	299.8
abvolts (cgs)	10^{8}	1	2.998×10^{10}
statvolts (cgs)	3.335×10^{-3}	3.335×10^{-11}	1

Electric Resistance

to obtain ↓ by Multiple number of →	ohms (mks)	abohms (cgs)	statohms (cgs)
ohms (mks)	1	10^{-9}	8.988×10^{11}
abohms (cgs)	10^{9}	1	8.988×10^{20}
statohms (cgs)	1.112×10^{-12}	1.112×10^{-21}	1

Problems

1.01. The electronic charge is -1.59×10^{-19} coulomb. If it is assumed that there are 8.48×10^{22} free electrons per cu. cm. of copper, find the average drift velocity of the free electrons in (a) a No. 4 A.W.G. copper wire carrying a current of 1 amp.; (b) a No. 4 A.W.G. copper wire carrying a current of 20 amp; (c) a No. 4 A.W.G. copper wire carrying a current of 500 amp.

1.02. Three identical resistors are connected first in series and then in parallel. What is the ratio of the resistance equivalent to the series and the resistance equivalent to the parallel arrangement?

1.03. A telegraph line 50 miles long extends between points A and B. There is a single fault on the line. If a battery of 200 volts is connected from A to ground, the potential of B, which is insulated, is 40 volts; if A is insulated, it requires a battery of 300 volts from B to ground to bring A to a potential of 40 volts with respect to ground. How far from A is the fault?

1.04. A battery having an emf of E volts and an internal resistance of R_1 ohms is connected to a variable resistance R_2. To what value should R_2 be adjusted to obtain the greatest possible power output from the battery?

1.05. The voltage across the terminals of an x-ray tube is 200 kv. and the current through the tube is 18 ma. Water is circulated in back of the anode (the electrode at the surface of which x-rays are generated) at the rate of 10 pt. per min.; the temperature differential between the input and output is 10.83°C. Calculate the approximate power of the x-ray beam that is emitted from the tube. Why is the result of this calculation likely to be inaccurate?

1.06. When a storage cell discharges 100 amp.-hr., 746,000 joules of chemical energy are transformed into electric energy; (a) how many watt-hours of electric energy are developed within the cell? (b) if the constant discharge current is 20 amp., what is the average electric power (in watts) developed? (c) if the average cell resistance during discharge is 0.015 ohm, what is the rate of dissipation of heat within the cell? (d) what is the electric power output of the cell? (e) what is the average value of the electric energy developed within the cell per coulomb of the electricity which flows through it? (f) what is the average electric power developed within the cell per ampere of the current flowing through it? (g) what is the average power per ampere dissipated as heat by the cell? (h) what is the average power output of the cell per ampere of the current through it?

1.07. A d-c meter movement deflects full-scale (50 divisions) when a current of 50 μa flows in its winding, which has a resistance of 500 ohms. Calculate the resistances of shunts that can be used to convert this meter to obtain full-scale deflections for 50 ma., 500 ma., and 5 amp. Calculate the resistances of multipliers which can be used with this meter to obtain full-scale deflections for 500 mv., 10 volts, and 500 volts.

1.08. Ten lamps, each having a resistance of 20 ohms, and a motor delivering 5 hp. with an efficiency of 80 per cent are connected in parallel. The combined load, across which there is an impressed voltage of 115 volts, is connected by means of two wires each having a resistance of 0.1 ohm to a generator having an internal resistance of 0.1 ohm. Calculate (a) the line current, (b) the emf of the generator, (c) the terminal voltage of the generator, (d) the power loss in the generator windings, (e) the power loss in the line, (f) the power input to the load.

1.09. What arrangement of 27 cells, each having an emf of 1.1 volts and an internal resistance of 2 ohms, will supply the greatest possible current to a resistance of 6 ohms?

1.10. Two resistors are connected in series to a 54-volt battery. A 50,000-ohm voltmeter connected across one resistor reads 13.0 volts; connected across the other it reads 22.8 volts. What are the magnitudes of the two resistances? (Neglect the internal resistance of the battery.)

1.11. Specify the interconnection of 24 dry cells (emf 1.4 volts, internal resistance 0.1 ohm) so that the maximum power is delivered to a load resistance of 0.6 ohm.

1.12. When a voltmeter is connected in series with 110 volts and a 35,000-ohm resistance, the deflection is 40 volts. When this voltmeter is connected in series with 220 volts and an unknown resistance R, the deflection is 10 volts. Calculate R.

1.13. A d-c generator with an internal resistance of 0.1 ohm and an emf of 50 volts is connected through a line having a resistance of 0.1 ohm to a battery having internal resistances and emfs which vary dependently as follows: emf 45 volts, resistance 0.25 ohm; emf 48 volts, resistance 0.20 ohm; emf 51 volts, resistance 0.15 ohm. Calculate the direction and magnitude of the current through the circuit for each of the three cases, the positive terminal of the battery being always connected to the positive terminal of the generator.

1.14. If n cells (emf e volts and resistance r ohms) are connected in p parallel-connected series of s cells each so that maximum power is delivered to a load resistance R, show that s and p are the integers nearest $\sqrt{nR/r}$ and $\sqrt{nr/R}$ and that the power delivered to R is $ne^2/4r$ watts.

1.15. A device for transforming alternating current into direct current is found to have the following properties:

D-c Output Voltage, volts	D-c Output Current, ma
300	50
275	100
250	150
225	200

Specify each part of a circuit that is equivalent to the device, *i.e.*, that has the same regulation.

1.16. Show that the load voltage V of a receiver to which a power P watts is delivered through a line of total resistance R ohms by a generator whose output voltage V_g is

What is the significance of the $\pm$ sign in this expression? What is the maximum power that can be delivered to the load if V_g and R are fixed? What is the ratio of the power lost in the line to the power delivered to the load for this condition?

1.17. The resistances of three wires BC, CA, AB of the same size and material are a, b, c, respectively. A wire of resistance d is connected to A and can make sliding contact with BC. What are the maximum and minimum resistances of the network for current entering one end, and leaving the other end, of the wire of resistance d?

1.18. If the charge q on an insulated metal plate varies with time according to the relation

$$= (1 - \varepsilon^{-2}) \qquad \text{coulomb}$$

what is the *current* flowing to the plate at $t = 1$ sec.?

1.19. Under what condition can the equation

of problem 18 be considered dimensionally homogeneous?

1.20. Is the equation

$$d$$

(a is constant acceleration) "correct"? Is it dimensionally homogeneous?

1.21. Show that the following relation among F (force), m (mass), v (velocity), and r (distance) is dimensionally homogeneous

1.22. In the following differential equation, L, R, C are constants having dimensions as yet unknown; the other symbols are q (electric charge), t (time), E (volts), and e (volts).

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = E + e \quad (E \text{ is constant; } e \text{ varies with time})$$

What dimensions must L, R, and C have if the relation is to be dimensionally homogeneous?

1.23. Assuming that the energy W, which is dissipated by a device of resistance R carrying a current I during a time interval t, is a function of R, I, t, develop the form of the function by dimensional analysis.

CHAPTER II

DIRECT-CURRENT CALCULATIONS

2.01. Introduction.—In this chapter electric networks will be considered in which the currents and emfs have values that are constant with respect to time or that, without appreciable error, may be assumed constant. Such currents are usually called *direct currents*, and such emfs are called *direct-current electromotive forces*, or d-c emfs.

Whenever a source of constant, or d-c, electromotive force is connected to a network by closing a switch, or whenever a change is made in the parameters of the network in which there are one or more such emfs, the currents in the various branches of the network change from one set of constant values to a new set of constant values, but this change usually takes place very quickly (in a second or less). The behavior of the network during this brief time interval will be considered in detail in a later chapter on *transient currents*.

2.02. One-, Two-, and Three-mesh Direct-current Networks. As explained in Sec. 1.14, Kirchhoff's laws for networks carrying direct currents may be formulated mathematically by assuming the existence in each mesh of a *mesh* current that has the same value in each branch of this mesh, and then writing down for each mesh an equation of the form

$$(2.01) \qquad R_{11}I_1 - R_{12}I_2 - R_{13}I_3 - \cdots = E_1$$

In this equation R_{12}, R_{13}, . . . are the resistances of the branches of this mesh, here designated 1, which are common respectively to meshes 2, 3, . . . , and R_{11} is the sum of the resistances of *all* the branches of this mesh 1, including any *outside* branch, say R_{10}. The emf E_1 is the algebraic sum, in the clockwise direction, of the generated emfs in the several branches of this mesh, including the emf in any outside branch. The group of equations of the form Eq. (2.01) will hereafter be referred to as the *mesh equations* for the network.

In the case of a single-mesh network, such as shown in Fig. 2.01, there is no need of employing double-subscript notation, provided one indicates by arrows the assumed positive sense of the
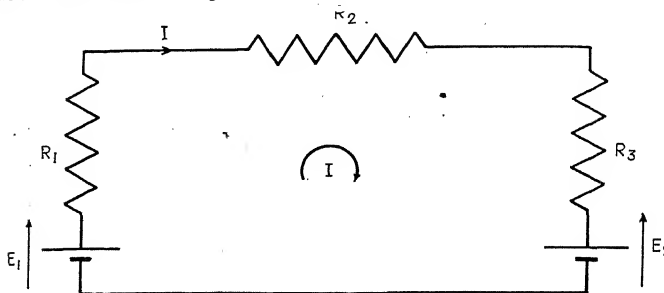

Fɪɢ. 2.01.

current and the assumed positive sense of each emf.   In the case of the single mesh depicted in the diagram, the mesh equation is

$$(2.02) \qquad (R_1 + R_2 + R_3)I = E_1 - E_2 \qquad \text{volts}$$

The solution of this equation for the current I, which in this case is the current in each of the resistances and through each of the sources of emf, is

$$(2.03) \qquad I = \qquad + \bar{R}_2 \qquad \text{amperes}$$

Consider next a two-mesh network, such as shown in Fig. 2.02. In this diagram, as in all subsequent diagrams for d-c calculations,
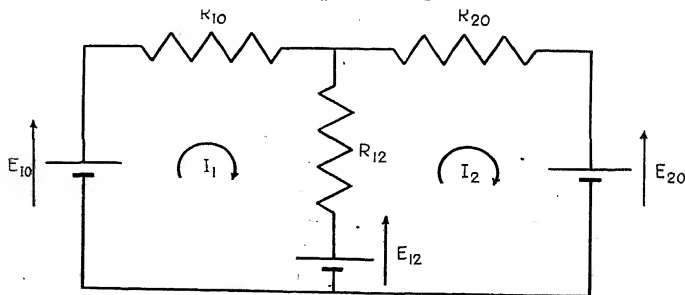
Fɪɢ. 2.02.

the double subscript attached to an E indicates simply the branch in which this emf is located, and the sequence of these subscripts

is of no significance.   The positive sense of each such emf will always be indicated by an arrow pointing in this positive sense.

For the network depicted in Fig. 2.02 the *mesh* resistances and the *mesh* emf of the two meshes are respectively

(2.04)
$$R_{11} = R_{10} + R_{12}$$
$$R_{22} = R_{20} + R_{12}$$

$$E_2 =$$

The two mesh equations for this network are then

(2.05)
$$R_{11}I_1 - R_{12}I_2 =$$
$$-R_{12}I_1 + R_{22}I_2 =$$

The solution of these two equations for the two mesh currents gives

(2.06)
$$I = \frac{R_{22}E_1 + R_{12}E_2}{R_{11}R_{22} - R_{12}^2}$$ amperes

These are also the branch currents in the outside branches of meshes 1 and 2 respectively.   The branch current in the common branch, in the same sense as the mesh current I_1, is

(2.07)
$$I_{12} = I_1 - I_2$$ amperes

The d-c mesh equations for *any three-mesh* network are

(2.08)
$$-R_{12}I_1 + R_{22}I_2 - R_{23}I_3 = E_2$$
$$-R_{13}I_1 - R_{23}I_2 + R_{33}I_3 = E_3$$

When there are no specific relations among the parameters or among the electromotive forces the solution of these three equations is most conveniently effected by the use of determinants, as follows.   Put

(2.09)
$$A_{11} = R_{22}R_{33} - R_{23}^2$$
$$A_{22} = R_{11}R_{33} - \qquad\qquad\qquad - R_{12}R_{23}$$
$$A_{33} = R_{11}R_{22} - \qquad\qquad A_{23} = R_{23}R_{11} -$$

Also put

(2.10)
$$D = A_{11}R_{11} + A_{12}R_{12} + A_{13}R_{13}$$

This quantity D is called the *determinant* formed by the resistances in, and the A's are called the cofactors of this determinant, corresponding to the elements that have the same subscripts. The solution for the three currents is then

$$I_1 = \frac{1}{D} \qquad\quad + A_{12}E_2 + A_{13}E_3)$$

(2.11) $$I_2 = \frac{1}{D}(A_{12}E_1 + \qquad\quad + A_{23}E_3) \qquad \text{amperes}$$

$$I_3 = \frac{1}{D} \qquad\quad + A_{23}E_2 + A_{33}E_3)$$

In Fig. 2.03 is shown diagrammatically a typical three-mesh network, in which there are three outside branches and with an
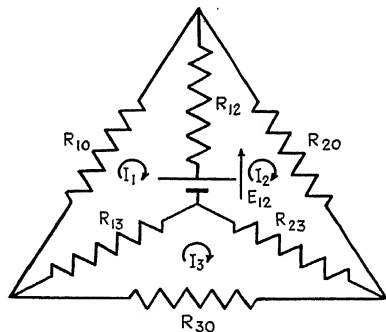
Fig. 2.03.

emf in one branch only. A Wheatstone's bridge may always be represented by such a diagram, and so may each section of a so-called *lattice-type filter*. The mesh resistances in this case are

(2.12)
$$R_{11} = R_{10} + R_{12} + R_{13}$$
$$R_{22} = R_{20} + R_{12} + R_{23} \qquad \text{ohms}$$
$$R_{33} = R_{30} + R_{13} + R_{23}$$

and the mesh emfs are

(2.13)
$$E_1 = -$$
$$E_2 = \qquad\qquad \text{volts}$$

For these particular conditions the mesh current in mesh 3, which is also the current in the outside branch of resistance R_{30}, is

(2.14) $$\qquad\qquad\qquad D \qquad\qquad\qquad \text{amperes}$$

where D has the value given by (2.10).   Note that if Fig. 2.03 represents a Wheatstone's bridge, and R_{30} represents the galvanometer, the null point $(I_3 = 0)$ is obtained when

(2.15) $$\frac{R_{10}}{R_{13}} = \frac{R_{20}}{R_{23}}$$

2.03. General Direct-current Networks; Network Theorems. The methods of solving d-c networks discussed in Sec. 2.02 can be extended to include as many meshes as there are in any network. The mechanics of the solution of an n-mesh network $(n > 3)$ differ from the mechanics of the solution of a three-mesh network only in algebraic complexity.   This complexity is minimized by the application of the rules for the manipulation of determinants, which are discussed in many textbooks of algebra.[1] Therefore, further discussion of the general theory of n-mesh d-c networks is omitted from this book.

There are several general *network theorems* that can be applied in order to reduce the labor of solving many d-c problems. Special examples of these theorems are stated and proved for three-mesh networks in this section; examples of their uses are presented in Sec. 2.06.

SUPERPOSITION THEOREM.   *Any mesh current I_h in an electrical network whose parameters are constants can be calculated by adding algebraically the currents I_h', I_h'' . . . that would be produced in the hth mesh by each emf of the original network acting alone.*   Thus I_2 of Eq. (2.11) is clearly the algebraic sum of

(2.16) $$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{amperes}$$

$$I_2''' = \frac{A_{23}}{D} E$$

and since E_1, E_2, E_3 are, in general, the algebraic sums of several emfs, each equation of Eq. (2.16) can, in general, be subdivided into several equations.   The currents I_2', I_2'', I_2''' . . . can usefully be designated as the *components* of the mesh current I_2. The applications of the superposition theorem of d-c networks

are simple and their usefulness is limited.   The theorem holds also for a-c networks and its general applications are extremely important.   This theorem is introduced here simply to call attention to the possibilities of the more general applications of it, which will be discussed in subsequent chapters on a-c and transient calculations.   Note that from the mathematical point of view the statement of the superposition theorem for d-c networks represents the recognition of the mesh equations (2.08) as *linear simultaneous equations*.   Thus this theorem is derived from the mathematical equations representing the networks, and then interpreted from the physical point of view.

RECIPROCAL THEOREM.   *The component of a mesh current I_h in the hth mesh attributable to an emf E'_m in an outside branch of the mth mesh is equal to the component of I_m that would be produced in the mth mesh if E'_m were transferred to the hth mesh.*   For the three-mesh circuit whose currents are listed in Eq. (2.11), put $E_1 = E_2 = 0$ in order to calculate the component I_1 produced in mesh 1 by E_3, in an outside branch of mesh 3, above

$$(2.17) \qquad\qquad I'_1 = \qquad\qquad\qquad \text{amperes}$$

Now transfer E'_3 to an outside branch of mesh 1, leaving meshes 2 and 3 with no emfs.   The component I'_3 produced in mesh 3 by an emf of magnitude E'_3 *placed in an outside branch of mesh* 1 is

$$(2.18) \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{amperes}$$

precisely the same as I'_1 calculated from Eq. (2.17), thus proving this special form of the reciprocal theorem.   A general form of the theorem, in which the emf E' is originally in a branch common to two meshes, say the mth and nth, and E' is put subsequently in a branch common to two other meshes, say the hth and kth, can be derived in a similar manner.   The student should prove in this case that the components of the *branch currents I_{mn}* and I_{hk} are equal.

THÉVENIN'S THEOREM.   *If two terminals A and B of a d-c network are to be connected to some external circuit, the network can be replaced by an emf E in series with a resistance R in order to calculate the current in the external circuit; E is the difference of potential between A and B when nothing external to the network is*

connected to them; R is the ratio of E to the current that flows if A and B are short-circuited. The resist-
ance R is called the *resistance looking into terminals A and B;* it is equal to the resistance that would be measured between A and B if all the emfs were zero. Proof of this theorem, for a special case, follows.

A typical three-mesh network is shown in Fig. 2.04. Consider the outside branch of mesh 1 to have a section between two terminals A and B that is closed by a jumper of no resistance. The mesh equations for this network are

Fig. 2.04.

$$(2.19) \quad \begin{aligned} R_{11}I_1' - R_{12}I_2' - R_{13}I_3' &= E_1 \\ -R_{12}I_1' + R_{22}I_2' - R_{23}I_3' &= E_2 \\ -R_{13}I_1' - R_{23}I_2' - \end{aligned}$$

and their solution for I_1' is

$$(2.20) \quad \frac{A_{11}E_1 + A_{12}E_2}{D} \quad \text{amperes}$$
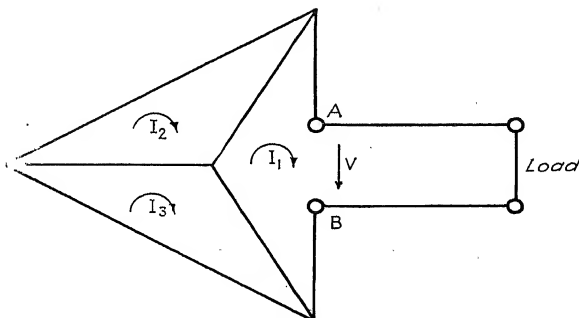
Fig. 2.05.

When the jumper between A and B is removed and an external branch, or *load,* is connected between A and B as indicated in Fig. 2.05, there will in general be a voltage drop (positive or negative) established from A to B. Designate this voltage drop by the symbol V and designate the resulting mesh currents by the same letters as in Fig. 2.05 but without the superscript ′.

The mesh equations for this resulting network are then

(2.21)
$$R_{11}I_1 - R_{12}I_2 - R_{13}I_3 = E_1 - V$$
$$-R_{12}I_1 + R_{22}I_2 - R_{23}I_3 = E_2$$
$$-R_{13}I_3 - R_{23}I_2 + R_{33}I_3 = E_3$$

and their solutions for I_1' is identical with their solution for I_1 except that $(E_1 - V)$ takes the place of E_1 in the numerator of
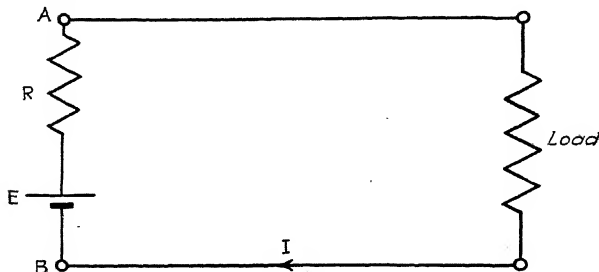

FIG. 2.06.

Eq. (2.20). Therefore, under these conditions the current in the load is

(2.22)
$$I = I' \quad \frac{A_{11}V}{D} \qquad \text{amperes}$$

Consider now the three-mesh network replaced by a single branch, or *source,* having a resistance R and an emf E in the sense from A to B, as shown in Fig. 2.06. The current produced in the load under these conditions is

(2.23)
$$I = \frac{E}{R} - \frac{V}{R} \qquad \text{amperes}$$

Therefore if R and E are so chosen that

(2.24)
$$R = \frac{D}{A_{11}} \qquad \text{ohms}$$

and

(2.25)
$$E = RI' \qquad \text{volts}$$

then the current I, which is the load current, will be identically the same whether the load is supplied by the single-branch source or by the three-mesh network.

From Eq. (2.23) it follows that E is equal to the value of V when there is no current in the load, which is the case when the terminals A and B are open-circuited.

In other words, the numerical value of E is the *open-circuit voltage* at the terminals A and B. From Eqs. (2.25) and (2.20) it follows that R is the ratio of this *open-circuit* voltage V to the current from A to B when the terminals are short-circuited by the jumper, which current is conveniently referred to as the *short-circuit* current from A to B.

COMPENSATION THEOREM. *If a resistance R be added to (in series with) an outside branch of the hth mesh of a network, the increase in the current I_k of any other mesh is equal to the product of (1) the number $(-R)$, (2) the mesh current I_h of the original network, (3) the ratio of the cofactor A'_{hk} and the determinant D' of the modified network.* Furthermore, if R is a small fraction of the mesh resistance R_{hh} to which it is added, an *approximate*

value of the increase in I_k $\quad -RI_h \dfrac{A_{hk}}{D}\Big)$ where A_{hk} and D and

calculated *for the original network.* The proof of this theorem for the addition of a resistance R to an outside branch of mesh 1 of a three-mesh network is given below. The original network is represented by

$$(2.08) \quad \begin{aligned} R_{11}I_1 - R_{12}I_2 - R_{13}I_3 &= E_1 \\ -R_{12}I_1 + R_{22}I_2 - R_{23}I_3 &= E_2 \\ -R_{13}I_1 - R_{22}I_2 + R_{33}I_3 &= E_3 \end{aligned}$$

and the current I_3 is

$$(2.11) \qquad I_3 = \frac{\cdot}{D} \qquad + \qquad + . \qquad \text{amperes}$$

the values of the A's and of D being those given in Eqs. (2.09) and (2.10). The addition of R to an outside branch of mesh 1 changes Eq. (2.08) to

$$(2.26) \quad \begin{aligned} (R_{11} + R)I'_1 - R_{12}I'_2 - R_{13}I'_3 &= E_1 \\ -R_{12}I'_1 + R_{22}I'_2 - R_{23}I'_3 &= E_2 \\ -R_{13}I'_1 - R_{23}I'_2 + R_{33}I'_3 &= E_3 \end{aligned}$$

in which the primed I's are the currents *after* the resistance R has been added. The solution for I'_3 is:

$$(2.27) \qquad\qquad \frac{\scriptstyle 1}{D} \qquad\qquad\qquad \text{amperes}$$

The A's and the D of Eq. (2.27) differ from the A's and the D of Eq. (2.11) because of the presence of R in the first term of the first equation of Eq. (2.26). Next consider three simultaneous equations

$$
\begin{array}{lll}
(a) & (R_{11} + R)I_1 - R_{12}I_2 - R_{13}I_3 = E_1 + RI_1 \\
(2.28) \quad (b) & \qquad -R_{12}I_1 + R_{22}I_2 - R_{23}I_3 = E_2 \\
(c) & \qquad -R_{13}I_1 - R_{23}I_2 + R_{33}I_3 = E_3
\end{array}
$$

From the physical point of view Eqs. (2.28) and (2.08) are equivalent; the currents are identical. From the mathematical point of view, two facts are noteworthy: (1) the terms RI_1 appearing on both sides of Eq. (2.28a) can be canceled, reducing Eq. (2.28) to Eq. (2.08) and thereby proving that the currents are identical; and (2) the determinant and cofactors of Eq. (2.28) are identical to those of Eq. (2.26). It follows from Eq. (2.28) using D', A'_{131} . . . from Eq. (2.26), that

$$
(2.29) \qquad = \frac{1}{D'} (A'_{13}(E_1 -
$$

Subtracting Eq. (2.29) from Eq. (2.27), the *increase* in I_3 caused by the introduction of R into an outside branch of mesh 1 is

$$
(2.30) \qquad\qquad\qquad\qquad\qquad\qquad \text{amperes}
$$

thus proving the theorem. Note that, if R is small enough so that

$$
(2.31) \qquad\qquad\qquad \frac{A'_{13}}{D'} \qquad \frac{A_1}{D}
$$

the increase in I can be calculated approximately in terms of the known quantities R, I_1, A_{13}, and D.

It is important to note that the theorems that are stated and proved in this section are special cases of more general theorems that can be applied to problems involving both transient and steady-state alternating currents in linear networks. The more general aspects of these theorems are discussed in subsequent chapters of this book.

2.04. T-section Attenuators.—A resistance attenuator is a four-terminal *passive** network of resistance elements whose

---

* A *passive network* is one in which there are no internal emfs. A network having emfs connected *within* it is commonly called an *active network*.

output voltage is a known, and often adjustable, fraction of the
input voltage.   One of the simplest attenuators is a potentiom-
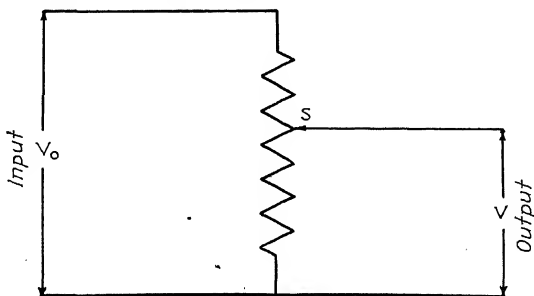eter, shown in Fig. 2.07.   Both the input and output currents

FIG. 2.07.

and the input and output resistances of this simple attenuator
vary with the adjustment of the slider or tap switch S.   Since
there are problems for which it is important to have equal and
constant input and output attenuator resistances, and constant
input current, the potentiometer is not universally applicable
as an attenuator.
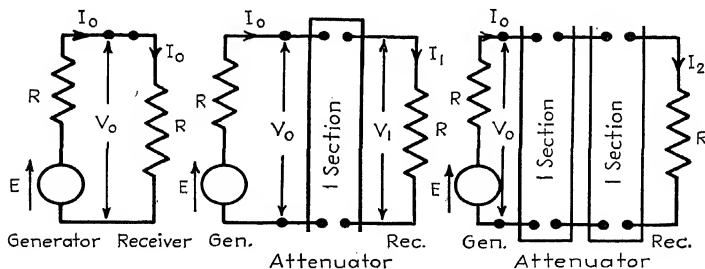
FIG. 2.08.

It is the purpose of this section to discuss one method of design-
ing a passive variable-resistance network whose input and output
resistances are equal and constant throughout the range of
attenuation for which the device is used.   If the input resistance
of the attenuator is constant, the current output *of the source*
remains constant.   Thus the particular problem to be considered
here is the design of one *section* of an attenuator, one or more of
which can be used as indicated in Fig. 2.08 to produce different

attenuations.   The attenuation can be expressed by ratios that, in the cases of 0, 1, and 2 sections shown in Fig. 2.08, have the relative values $\dfrac{I_0}{I_0}, \dfrac{I_1}{I_0}, \dfrac{I_2}{I_0}$.   Each section can be designed in a variety of forms, of which three are shown in Fig. 2.09.   Only the symmetrical T section is considered in the discussion which follows.

Symmetrical T-Section   Symmetrical π-Section   Symmetrical Bridge or Lattice Section

Fig. 2.09.

Fig. 2.10.—$\dfrac{V_0}{I_0} = \dfrac{V_1}{I_1} = R.$
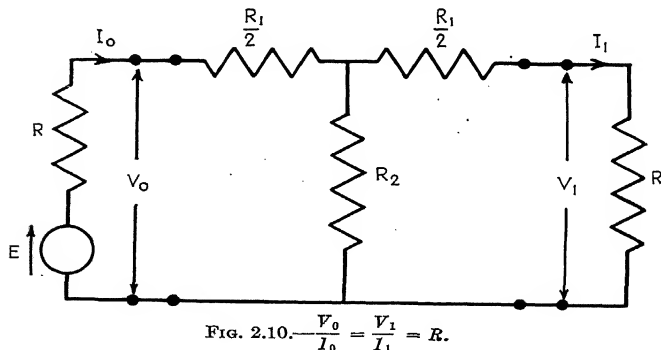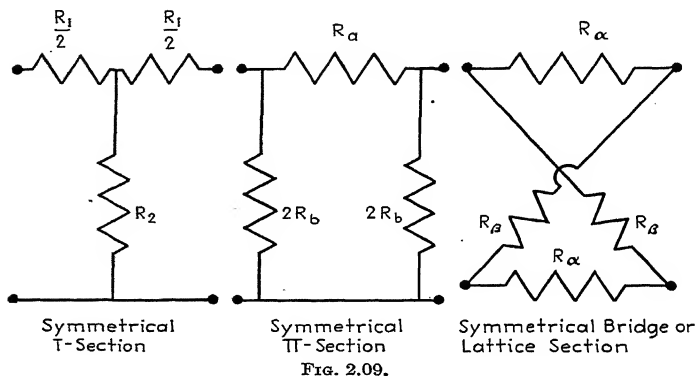
From Fig. 2.08 it is clear that the elements of the symmetrical T section must be chosen so that the conditions of Fig. 2.10 are satisfied.   Solution of the mesh equations and substitution of R for V_0/I_0 show that R_1 and R_2 must have values that satisfy

(2.32) · $R = \tfrac{1}{2}[R_1(R_1 + 4R_2)]^{\frac{1}{2}}$ ohms

An infinity of pairs of values of R_1 and R_2 will satisfy Eq. (2.32). A pair of unique values requires the mathematical statement of some property of the network other than that stated in Eq. (2.32). One such useful property is the amount of attenuation produced by one section. If the ratio $I_1/I_0(< 1)$ is assigned the symbol A, the mesh currents for Fig. 2.10 show that

$$(2.33) \qquad A \equiv \frac{I_1}{I_0} = \frac{R_2}{(R_2/2) + R_2 + R} \qquad \text{(dimensionless)}$$

Solving Eqs. (2.32) and (2.33) simultaneously for R_1 and R_2 in terms of R and A

$$(2.34) \qquad\qquad = 2R\,\frac{1-A}{1+A} \qquad\qquad \text{ohms}$$

$$(2.35) \qquad\qquad R_2 = 2R \qquad\qquad \text{ohms}$$

Note that if the receiver R is replaced by a second T section and R is connected to the output of the second section, the ratio of the receiver current I_2 to the input current I_0 is A^2. As more sections are added, the attenuation increases so that the ratios of output current to input current for 0, 1, 2, 3, 4 . . . n sections are A^0, A^1, A^2, A^3, A^4, . . . A^n. The attenuation is often expressed in dimensionless units called *decibels** (db) defined in terms of the ratio of input current I_0 to output current I_n, or of input voltage V_0 to output voltage V_n, or of input power P_0 to output power P_n as follows:

$$\text{db loss} = 20\,\log_{10}\frac{I_0}{I}$$

$$(2.36) \qquad \text{db loss} = 20\,\log_{10}\frac{V_0}{V}$$

$$\text{db loss} = 10\,\log_{10}\frac{P_0}{P}$$

In the specific case here considered, the input and output resistances are equal; all three calculations of *decibel loss* are, therefore, equal. If losses in other circuits are calculated in decibels, care must be taken that the current, voltage, or power ratios consist of comparable quantities; otherwise the results will be meaning-

---

* Attenuation can also be expressed as $\frac{1}{2}\log_\epsilon \frac{P_0}{P}$; the unit is called the *neper;* 1 neper is equal to 8.686 decibels.

less, or they may be ambiguous. *The accepted definition of decibel loss is that involving the ratio of two powers.* Further consideration of the use of these units for measuring losses (or gains in transformers or amplifiers) is reserved for subsequent chapters on alternating currents. Note that the decibel loss of one attenuator can be added to the decibel loss of a second attenuator in order to obtain the total loss caused by both devices.

If R_1 and R_2 of Fig. 2.10 are known, the load resistance R that will cause the input resistance of the attenuator also to be R
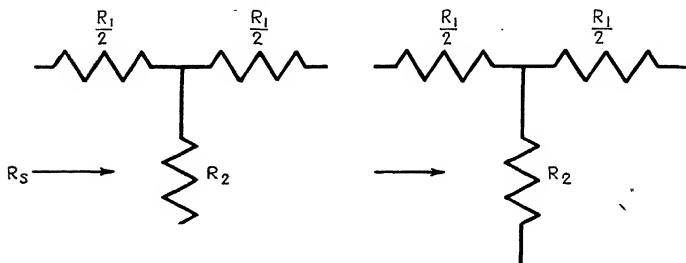
Fig. 2.11.

can be calculated from Eq. (2.32) and the attenuation ratio per section $\frac{I_1}{I_0}$ can be calculated from Eq. (2.33). The resistance R is called the *characteristic resistance* of the T section R_1, R_2.

Note from Fig. 2.11 that the *short-circuit resistance R_s* of one section of the attenuator is

$$(2.37) \qquad R_s = \frac{R_1}{2} + \frac{R_1 R_2}{R_1 + 2R_2} \qquad \text{ohms}$$

and that the *open-circuit resistance R_0* of one section is

$$(2.38) \qquad R_0 = \frac{R_1}{2} + R_2 \qquad \text{ohms}$$

The square root of the product of Eqs. (2.37) and (2.38) reduces to Eq. (2.32) so that

$$R = \sqrt{R_0 R_s} \qquad \text{ohms}$$

The last three expressions have important general implications.

It is evident that any four-terminal passive network can be replaced by an equivalent T section. The equivalent T, using a particular pair of input terminals and the other pair as output terminals, can be calculated from measurements of the input resistance (1) with the output terminals short-circuited (R_s), and (2) with the output terminals open-circuited (R_0). The elements $\left(\dfrac{R_1}{2},\ R_2\right)$ of the symmetrical T section are

$$(2.40) \qquad \frac{R_1}{2} = R_0 - \qquad\qquad \text{ohms}$$

$$(2.41) \qquad\qquad = \sqrt{R_0(} \qquad\qquad \text{ohms}$$

These values can be obtained by solving Eqs. (2.37) and (2.38) simultaneously.

2.05. Two-wire Transmission Line with Uniformly Distributed Resistance and Leakage.—The notation and methods discussed in Sec. 2.04 can be modified and used to analyze the
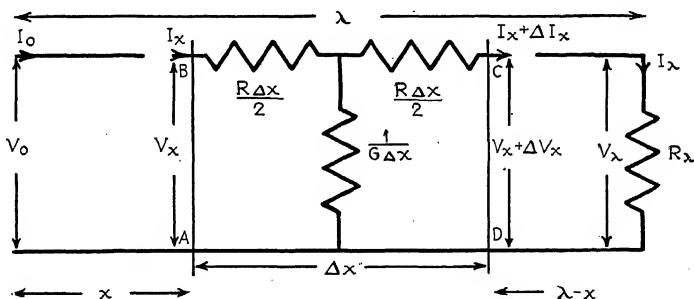
Fig. 2.12.

problem of two parallel conductors, each of resistance $R/2$ ohms per unit length, separated by a leaky insulator whose conductance per unit length of line is G mhos. Such a line is shown in Fig. 2.12. Its length is λ meters; distance along the line (x) is measured from the *sending* end at which a voltage V_0 is applied. The line is terminated at $x = \lambda$ by a resistance R_λ. Assume that a short (not infinitesimally short) length Δx of the line at a distance x from the sending end of the line can be approximately represented by the T section shown in Fig. 2.12. The characteristic

resistance of this T section is, from Eq. (2.32)

$$(2.42) \qquad R'_c = \frac{1}{2}\left[R^2(\Delta x)^2 + 4\frac{R}{G} \right]^{\frac{1}{2}} \qquad \text{ohms}$$

Now assume that $R_\lambda = R'_c$; it follows at once from Eq. (2.33) that the ratio of the output current to the input current of any section corresponding to a length Δx of the line is

$$(2.43) \quad A =$$

In a length x of the line there are $\frac{x}{\Delta x}$ sections. Therefore, the current I'_x in either wire at a distance x from the sending end, when the line is terminated in R_c at λ, is approximately

$$I'_0 \left\{ \frac{RG(\Delta x)^2}{2} + 1 + \frac{G\,\Delta x}{2}\left[R^2(\Delta x)^2 + 4\frac{R}{G} \right]^{\frac{1}{2}} \right\}$$

This result is approximate because the line has been assumed to consist of $\frac{\lambda}{\Delta x}$ discrete T sections. An exact result can be obtained by evaluating Eq. (2.44) when $\Delta x \to 0$, if second-order terms in Δx are neglected.

$$(2.45) \qquad I'_x =$$

Now put $\Delta y = \sqrt{RG}\,\Delta x$ so that Eq. (2.45) becomes

$$(2.46) \quad I'_x = \lim_{} I'_0\left[(1 + \Delta y)\frac{1}{\Delta y} \right]^{-}$$

Thus the current in a two-wire transmission line with uniformly distributed resistance and leakage decreases exponentially with distance measured from the sending end *if it is terminated by its characteristic resistance R_c*, which is

$$(2.47) \qquad\qquad\qquad\qquad\qquad\qquad \text{ohms}$$

The line voltage can be calculated in a similar manner; the result is

$$(2.48) \qquad\qquad V'_x = \qquad\qquad\qquad \text{volts}$$

if the line is terminated by its characteristic resistance .

If the terminating resistance R_λ is different from R_c, the increase in current I_x'' caused by the change in resistance from R_c to R_λ can be calculated according to the compensation theorem by considering the effect of an emf $(-RI_x)$, where $R = R_\lambda - R_c$, connected in series with R_c at the receiving end of the line. Such an emf would produce a current I_x''

$$(2.49) \qquad I_x'' = \left(- \frac{R}{2R_c} I_0 \varepsilon^- \right.$$

in the line at a distance x from the "sending" end [$(\lambda - x)$ from the "receiving" end], if the terminating resistance at $x = 0$ be R_c. It follows, therefore, that the current at any point in the transmission line is in general of the form

$$(2.50) \qquad\qquad\qquad\qquad\qquad\qquad \text{amperes}$$

in which I_1 and I_2 depend upon the input and output circuits. Similarly

$$(2.51) \qquad\qquad V_x = \qquad\qquad + \qquad\qquad\qquad \text{volts}$$

in which V_1 and V_2 depend also upon boundary conditions; only two of the coefficients I_1, I_2, V_1, V_2 are independent. As a specific example of boundary conditions, suppose that the input voltage V_0, the length of the line λ, and the terminating resistance R_λ are known. The coefficients of Eqs. (2.50) and (2.51) can be shown to be

$$(2.52) \qquad V_1 = \frac{(R_\lambda}{(R_\lambda + R_c)\varepsilon^{\sqrt{RG}\,\lambda} + (R_\lambda}$$

$$(2.53) \qquad\qquad \frac{(R_\lambda + j}{(R_\lambda +}$$

$$(2.54) \qquad I_1 = -\frac{V_1}{R_c}$$

$$(2.55)$$

The student should derive these results for practice. Note that, if the line is infinitely long, $V_1 = I_1 = 0$ and the input resistance is R_c.

The problem of the d-c line can also be solved by deriving two differential equations involving x as the independent variable and V_x and I_x as the dependent variables and by solving these equations. Using the notation indicated in Fig. 2.12 and Kirchhoff's laws, note the following relations:

From Kirchhoff's first law applied to junction point C, Fig. 2.12

(2.56) $I_x = I_x + \Delta i$

From Kirchhoff's second law applied around $ABCDEA$, Fig. 2.12

(2.57) $V_x = V_x + \Delta V_x$ -

The limits as $\Delta x \cdot \quad$, are

(2.58) $$\overline{dx}$$

(2.59) $$\overline{dx} = -RI_x$$

By taking the derivatives with respect to x of Eqs. (2.58) and (2.59) and rearranging the four equations, the following second-order equations are obtained:

(2.60) $$\frac{d^2 V_x}{dx^2}$$

(2.61) $$\overline{dx^2} = RGI_x$$

Substitution of

(2.51) $V_x = \qquad\qquad\qquad\qquad$ volts

and

(2.50)   . $I_x = I_1 \varepsilon^{\vee \text{RG} x} + I_2 \varepsilon^{-\vee \text{RG} x}$ amperes

in Eqs. (2.60) and (2.61) shows that they satisfy the derived equations; V_1, V_2, I_1, I_2 are arbitrary (integration) constants that must be evaluated from the boundary conditions of each problem. Equations (2.52) to (2.55) are illustrations of the

procedure. Thus both methods of derivation of the current and voltage along the line lead — as it should be expected — to the same results.

2.06. Illustrative Problems.
Problem 1.

a. Calculate the currents I_{BA} and I_{GH} in the circuit shown in Fig. 2.13.
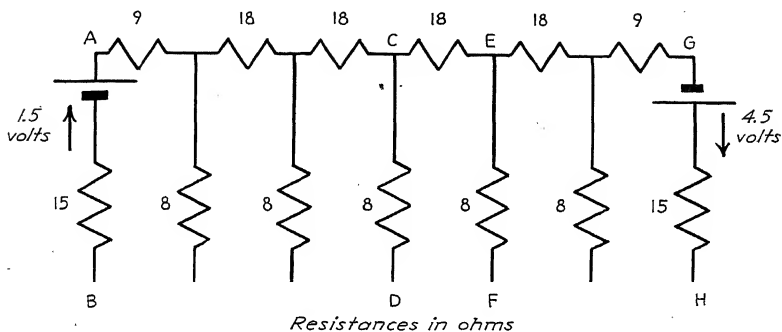
Resistances in ohms
FIG. 2.13.

b. If the resistance in the branch GH were increased to 15.25 ohms, what would the currents I'_{BA} and I'_{GH} be?

c. If the two emfs shown in the diagram were zero, and an emf $E_{FE} = 3$ volts were inserted in the branch EF, what would the current I''_{BA} be?

d. If three networks like that shown in Fig. 2.13 were connected as indicated by Fig. 2.14, what would the ammeter current be?
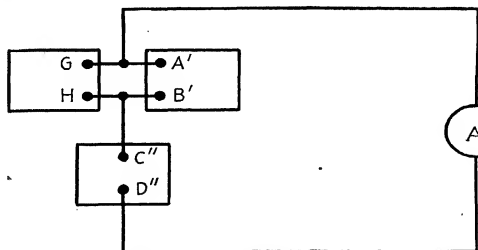
FIG. 2.14.

Solution.—The network between AB and GH consists of 5 symmetrical T sections for which $R_1 = 18$ ohms and $R_2 = 8$ ohms. From Eq. (2.32) the characteristic resistance R_c of such a section is

$$R_c = \tfrac{1}{2}[18(18 + 32)]^{\frac{1}{2}} = 15 \qquad \text{ohms}$$

so that the T sections are terminated by the characteristic resistance at both ends. The attenuation ratio for one section is, from Eq. (2.33),

$$A = \frac{8}{9 + 8 + 15} = \frac{1}{4} \qquad \text{per section}$$

The current I_{BA} can be considered, according to the superposition theorem, as the sum of two components: (1) I_{BA}^{*}, the current produced in BA by the 1.5-volt emf in BA *alone;* and (2) I_{BA}^{**}, the current produced in BA by the 4.5-volt emf in GH *alone.* The first component I_{BA}^{*} is the ratio of 1.5 volts to the total resistance, 15 ohms in BA and 15 ohms for the remainder of the network. Thus

$$\qquad = 0.05 \qquad \text{amp.}$$

The second component I_{BA}^{**} is the current through the 4.5-volt emf when it acts *alone*

$$I_{GH}^{*} = 0.15 \qquad \text{amp.}$$

multiplied by the attenuation ratio $A = \frac{1}{4}$ raised to the fifth power (there are five sections)

$$I_{BA}^{**} = 0.15(\tfrac{1}{4})^5 = 0.000146 \qquad \text{amp.}$$

Therefore

$$I_{BA} = I_{BA}^{*} + I_{BA}^{**} = 0.05015 \qquad \text{amp.}$$

Calculating I_{GH} in a similar manner the result is

$$I_{GH} = 0.15005 \qquad \text{amp.}$$

These are the answers to part *a.* Note that these calculated answers are expressed by a greater number of significant figures than conventional measuring methods would provide. Answers of 50 ma. and 150 ma. for I_{AB} and I_{GH} would be considered accurate enough for most purposes. In other words, the current produced at one end of the network by the emf at the other end is practically insignificant compared to the current at one end produced by the emf at that end.

According to the compensation theorem, the effects upon I_{BA} and I_{GH} of changing the resistance in GH from 15 to 15.25 is *approximately* equal to the change produced by an emf $(-0.25)(I_{GH})$ connected in series with GH. Thus

$$\Delta I_{GH} = -(0.25)(0.15) = -0.00125 \qquad \text{amp.}$$
$$\Delta I_{BA} = (-0.00125)(\tfrac{1}{4})^5 = -0.00000122 \qquad \text{amp.}$$

In answer to part *b*, then, $I_{BA}' = I_{BA}$, the change being practically negligible, while $I_{GH}' = 0.1488$ amp., *i.e.*, it is reduced by 1.25 ma.

Part *c* is most easily answered by applying the reciprocal theorem. If an emf of 3 volts in branch EF produced a current $I_{AB}^{\#}$ in BA, this current is equal, according to the reciprocal theorem, to the current $I_{EF}^{\#}$ pro-

duced in EF by an emf of 3 volts in BA.   Since

$$I_{EF}^{\#} = \tfrac{3}{30}[(\tfrac{1}{3})^3 - (\tfrac{1}{4})^4] = 0.00117 \qquad \text{amp.}$$

it follows that the answer to part c is

$$I_{AB}^{\#} = 0.00117 \qquad \text{amp.}$$

The simplest calculations of the ammeter current of part d involve first the determinations of the three series circuits which are equivalent, according to Thévenin's theorem, to the three networks.   The open-circuit voltage rise E_{GH} of the network shown in Fig. 2.13 is

$$E_{GH} = 4.5 - 15I_{GH} = 2.2494 \qquad \text{volts}$$

If the emfs within the network are zero, the resistance R_{GH} looking into GH is

$$R_{GH} = \tfrac{15}{2} = 7.5 \qquad \text{ohms}$$

Similarly, the open-circuit voltage rise $E'_{B'A'}$ and the resistance $R'_{A'B'}$ of the second network are

$$1.5 - 15I_{BA} = 0.7482 \qquad \text{volt}$$

and

$$7.5 \qquad \text{ohms}$$

The open-circuit voltage rise $E''_{C''D''}$ of the third network is the voltage drop through the 8-ohm resistor between C and D (Fig. 2.13).   The current

9 ohms   9 ohms

C''

15 ohms   8 ohms   15 ohms

D''
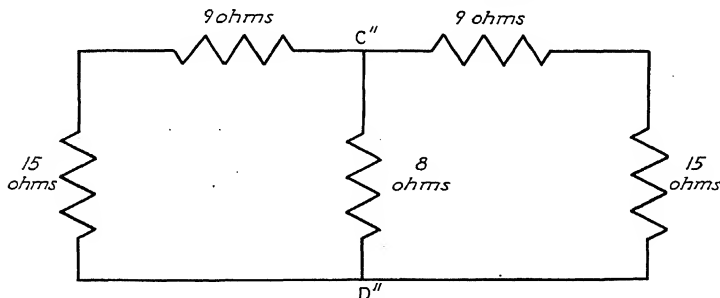
FIG. 2.15.

through this resistor is the difference between the currents produced by the 4.5-volt emf and the 1.5-volt emf

$$E''_{C''D''} = 8\left[0.15004\left(\frac{1}{4^2} - \frac{1}{4^3}\right) - 0.05012\left(\frac{1}{4^2} - \frac{1}{4^3}\right) \right] = 0.03747 \qquad \text{volts}$$

The resistance $R''_{C''D''}$ is the resistance between the terminals C and D in Fig. 2.15.   Therefore, the resistance $R''_{C''D''}$ is

$$R'' \qquad \tfrac{1}{8} - \frac{1}{\ } \qquad 4.8 \qquad \text{ohms}$$

Thus all three networks and the ammeter shown in Fig. 2.14 can be considered as being equivalent to the simple two-mesh circuit shown in Fig. 2.16. The circuit shown in Fig. 2.16 can be solved as a two-mesh network using
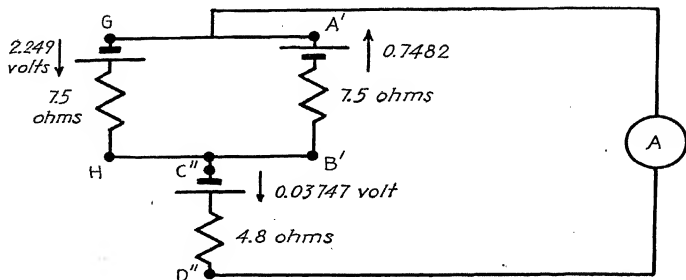
Fig. 2.16.

Eqs. (2.06), or the entire circuit except the ammeter can be reduced to a single emf E and resistance R by means of Thévenin's theorem. The open-circuit emf of the mesh $HGB'A'$ is $2.2494 - (0.1932)(7.5) = 0.7506$
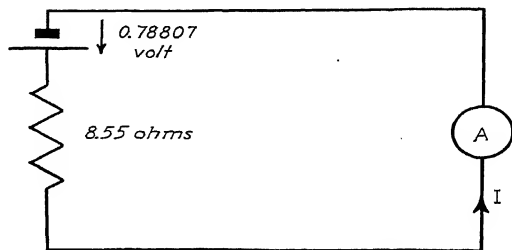

Fig. 2.17.

volts (negative terminal up, Fig. 2.16); the resistance is 3.75 ohms. Thus the circuit reduces to that shown in Fig. 2.17 so that, in answer to part d, the ammeter current I is

$$I = \frac{0.78807}{8.55} = 0.0922 \quad \text{amp.}$$

Problem 2.

A concentric cable consists of a single No. 20 A.W.G. copper wire along the axis of the cable, insulation around the wire, and a lead sheath; the sheath is used as the second conductor. The combined resistance of the wire and sheath is 0.04 ohm per m.; the conductance per unit length of the insulation is 3.42×10^{-4} μmhos per m. A piece of this cable 10 miles long is connected at one end to a 500-volt battery. Calculate the battery current, and the current and voltage at the other end of the cable, when the

latter is (*a*) short-circuited, (*b*) open-circuited, and (*c*) connected to a 10,000-ohm resistor.

Solution.—First calculate the attenuation factor:

$$= \sqrt{(0.04)(3.42 \times 10^{-10})} = 0.370 \times 10^{-5}.$$

the characteristic resistance $R_c = \sqrt{\dfrac{R}{G}} = \sqrt{\dfrac{0.04}{3.42 \times 10^{-10}}}$ 1.081×10^4 ohms, and the length of the line in meters $\lambda = 16{,}093.4$ meters.   Next calculate the exponential terms which will be required for the solution [see Eqs. (2.50 to 2.55)].

$$\sqrt{RG}\,\lambda = \varepsilon^{0.0595} = 1.0613$$
$$= \varepsilon^{-0.0595} = 0.9423$$

Part a.   Line Short-circuited.—The voltage and terminating resistance at the short-circuited end are zero.   Combining Eqs. (2.50), (2.52), (2.53), (2.54), (2.55) and putting $x = \lambda$.

$$I_{\lambda s} = -\frac{V_0}{R_c}\frac{-1}{\varepsilon^{\sqrt{RG}\,\lambda} - \varepsilon^{-\sqrt{RG}\,\lambda}} + \frac{V_0}{R_c}\frac{1}{\varepsilon^{\sqrt{RG}\,\lambda} - \varepsilon^{-\sqrt{RG}\,\lambda}}$$
$$I_{\lambda s} = 2\frac{500}{1.068 \times 10^4}\frac{1}{1.0613 - 0.9423} = 0.787 \qquad \text{amp.}$$

The input current I_{0s} is calculated in a similar manner, putting $x = 0$.

$$I_{0s} = -\frac{V_0}{R_c}\frac{-\varepsilon^{-\sqrt{RG}\,\lambda}}{\varepsilon^{\sqrt{RG}\,\lambda} - \varepsilon^{-\sqrt{RG}\,\lambda}} + \frac{V_0}{R_c}\frac{\varepsilon^{\sqrt{RG}\,\lambda}}{\varepsilon^{\sqrt{RG}\,\lambda} - \varepsilon^{-\sqrt{RG}\,\lambda}}$$
$$I_{0s} = \tfrac{1}{2}I_{\lambda s}(1.0613 - 0.9423) = 0.788 \qquad \text{amp.}$$

Thus about 1 ma. is lost through the insulation.

Part b.   Line Open-circuited.—The current at $x = \lambda$ is zero, and the terminating resistance is infinite.   From Eqs. (2.52) and (2.53) note that

$$\lim_{R_\lambda \to \infty} V_1 = \frac{\varepsilon^{-\sqrt{RG}\,\lambda}}{\varepsilon^{\sqrt{RG}\,\lambda} + \varepsilon^{-\sqrt{RG}\,\lambda}}$$
$$\lim_{R_\lambda \to \infty} V_2 = \frac{\varepsilon^{\sqrt{RG}\,\lambda}}{\varepsilon^{\sqrt{RG}\,\lambda} + \varepsilon^{-\sqrt{RG}\,\lambda}}$$

Therefore the voltage at the open-circuited end of the cable is

$$V_{\lambda 0} = \frac{V_0}{\varepsilon^{\sqrt{RG}\,\lambda} + \varepsilon^{-\sqrt{RG}\,\lambda}} + \frac{V_0}{\varepsilon^{\sqrt{RG}\,\lambda} + \varepsilon^{-\sqrt{RG}\,\lambda}}$$
$$V_{\lambda 0} = \frac{2V_0}{2.0036} = 499.1 \qquad \text{volts}$$

and the current through the battery is

$$I_{00} = \frac{V_0}{R_c}\frac{\varepsilon^{\sqrt{RG}\,\lambda} - \varepsilon^{-\sqrt{RG}\,\lambda}}{\varepsilon^{\sqrt{RG}\,\lambda} + \varepsilon^{-\sqrt{RG}\,\lambda}} = 0.00275 \qquad \text{amp.}$$

Part c.   Terminating Resistance 10,000 *Ohms.*—In this case the coefficients are those given in Eqs. (2.52) to (2.55):

$$V_1 = \frac{(-810)\varepsilon^{-\sqrt{RG}\,\lambda}V_0}{20{,}810\varepsilon^{\sqrt{RG}\,\lambda} - 810\varepsilon^{-\sqrt{RG}\,\lambda}} = -0.038 V_0 \varepsilon^{-\sqrt{RG}\,\lambda}$$

$$V_2 = \frac{20{,}810\varepsilon^{\sqrt{RG}\,\lambda}V_0}{20{,}810\varepsilon^{\sqrt{RG}\,\lambda} - 810\varepsilon^{-\sqrt{RG}\,\lambda}} = 0.976 V_0 \varepsilon^{\sqrt{RG}\,\lambda}$$

$$I_1 = 0.038 \frac{V_0}{R_c} \varepsilon^{-\sqrt{RG}\,\lambda}$$

$$I_2 = 0.976 \frac{V_0}{R_c} \varepsilon^{\sqrt{RG}\,\lambda}$$

Therefore the output current and voltage are

$$I_\lambda = \frac{500}{10{,}810}(0.038 + 0.976) = 0.0469 \qquad \text{amp.}$$

$$V_\lambda = 500(-0.038 + 0.976) = 469 \qquad \text{volts}$$

Note that $\frac{V_\lambda}{I_\lambda} = 10{,}000$ ohms, thus checking the numerical calculations. The input (battery) current is

$$I_0 = \frac{V_0}{R_c}(0.038\varepsilon^{-}$$

$$I_0 = 0.0496 \qquad \text{amp.}$$

Note that, if the line were terminated by a resistance $R_\lambda = R_c = 10{,}810$ ohms, the coefficient V_1 would be zero and the input current would be $\frac{V_0}{R_c} = 0.04625$ amp.   When the load resistance is less than R_c, as in parts *a* and *c*, the input current is greater than 0.04625 amp., while if the load resistance is greater than R_c, as in part *b*, the input current is less than 0.04625 amp.

Comment.—An experienced solver of problems such as this, who wanted approximate answers, would proceed as follows.   Note that in part *a* the resistance corresponding to the leakage resistance through the insulation cannot be less than $\frac{1}{(16{,}093)(3.42 \times 10^{-10})} = 184{,}000$ ohms, whereas, the resistance of the wire and sheath is $(16093)(0.04) = 643.7$ ohms.   The current in either end is therefore approximately $\frac{500}{644} = 0.78$ amp.

The input current for part *b* is approximately $\frac{500}{184{,}000}$ 0.0027 amp. and the voltage at the open-circuited end is approximately

$$500 - \tfrac{1}{2}(0.0027)(644) = 499.1 \qquad \text{volts}$$

The battery current for part *c* is approximately $\frac{500}{}$ 0.047 amp. and the voltage across the terminating resistor is therefore

$$(0.047)(10{,}000) = 470 \qquad \text{volts}$$

Bibliography

1. *A College Algebra;* H. B. Fine; Chap. XXXI, Ginn and Company, Boston, 1904.

Problems

2.01. Three dry cells, all having open-circuit terminal voltages of 1.5 volts, are tested by means of a short-circuit ammeter. The readings are 30 amp., 35 amp., and 38 amp. Assuming that the internal resistances of the cells are independent of the current through the cells, calculate the maximum power that the three cells can deliver when they are connected in parallel.

2.02. What is the current in R when it is adjusted so that the current in A is one-half the current in R? What is the current in R when it is adjusted so that the current through the battery is 10 amps.?
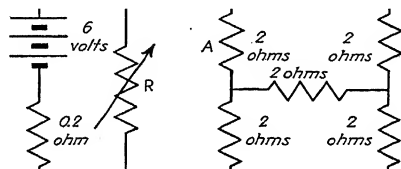
PROB. 2.02.

2.03. The voltages at each end of a 10-mile section of an electric railway are 600 volts and 575 volts. The total circuit resistance is 0.08 ohm per mile. A locomotive, through which a current of 1,000 amp. is flowing, is located at that point for which the voltage across it is minimum. How far is the locomotive from the 600-volt generator and how much current does each generator supply?

2.04. If 12 one-ohm resistors form the edges of a cube, what is the resistance between opposite corners?

2.05. Two cells having emfs E_1 and E_2, and resistances R_1 and R_2, are connected in parallel to a wire of resistance R. Show that the current in the wire is

$$\frac{E_1R_2 + E_2R_1}{R_1R + R_2R + R_1R_2}$$

and calculate the power output of each cell.

2.06. When S is open A reads 3 amp.; when S is closed A reads 1.5 amp. Calculate R_1, R_2, and the currents in both cases.
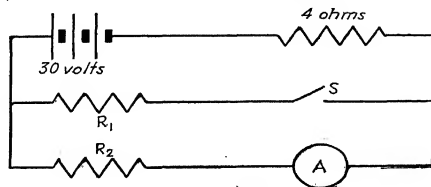
PROB. 2.06.

2.07. When a 16,000-ohm voltmeter is connected from one side, A, of a 125-volt circuit to ground, the meter reading is 2.6 volts. Calculate the insulation resistances from the second wire, B, to ground assuming that the insulation resistance from A to ground is (a) infinite, (b) equal to the resistance from B to ground, and (c) one-tenth of the resistance from B to ground.

2.08. A 250-volt generator is connected by means of a line 600 ft. long to a load operating at 242 volts, 300 amp. Calculate the resistance of the line, the power loss in the line, the efficiency of transmission of power, and the load that could be supplied by this line if the input voltage were 125 volts and the efficiency equal to that calculated for 250 volts.

2.09. Six wires of resistance r form the edges of a regular tetrahedron. Find the resistances of the network (a) between two apexes and (b) between the midpoints of opposite sides.

2.10. In the circuit shown in the figure below, A is a storage battery with an emf of 20 volts and an internal (series) resistance of 0.5 ohm; B is a storage battery with an emf of 22 volts and an internal (series) resistance of 1 ohm. (a) What current will flow in the 5-ohm resistance when the switch K is closed? (b) What current would flow in the 5-ohm resistance if the circuit was opened at K? (c) Are the powers supplied by A in the two cases equal? Explain.
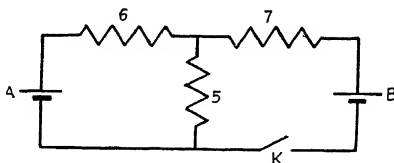
PROB. 2.10

2.11. If each point of a group of n points is joined to each of the other points of the group by a wire of resistance r, and two of the points are connected to the terminals of an emf E of which the internal resistance is R, show that the current in the *wire* joining these two points is:

$$\frac{2E}{2r + nR}$$

2.12. In the three-mesh circuit shown in the figure on page 71, the following are known:

$$
\begin{aligned}
E_1 &= 20 \text{ volts} & R_{23} &= 25 \text{ ohms} \\
E_2 &= 15 \text{ volts} & R_{30} &= 10 \text{ ohms} \\
R_{10} &= 6 \text{ ohms} & R &= 10 \text{ ohms} \\
R_{20} &= 10 \text{ ohms}
\end{aligned}
$$

R_{13} is a slide wire whose total resistance is 30 ohms. Find the proper

position of the slider A so that the current through the galvanometer shall be: (a) zero; (b) 100 ma.
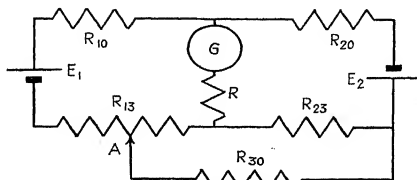
PROB. 2.12.

2.13. A branch of an interurban railway system is 8 miles long. At one end is a substation, A, which maintains a voltage 650 volts between the track and the trolley wire. At the other end is a storage battery substation, B, of negligible resistance, at which the voltage is 580 volts. Two cars are on the section, one following 1 mile behind the other, each drawing 50 amp. The trolley wire is No. 0000 A.W.G.; the combined resistance of the rail is 0.05 ohm per mile. (a) Locate the cars when the storage battery is neither charging nor discharging.

2.14. A wire of length $4a$ is formed into a square. Two other lengths of the same kind of wire form diagonals of the square and are joined at their intersection. A current I enters at the intersection of the diagonals and leaves at one of the corners of the square. Find the current in each part of the network in terms of I. Determine the length of wire which has the same resistance as the network.

2.15. Three trolley cars are operating simultaneously on a branch of an electric railway system. Car A is 1 mile, car B 3 miles, car C 4 miles from the powerhouse. The powerhouse terminal voltage is 600 volts. The resistance of the trolley wire is 0.25 ohm per mile. The resistance of each rail of the track is 0.1 ohm per mile. The rails are connected together at the powerhouse. The total resistance of the armature and series resistance of each car is 0.1 ohm. The back emf of car A is 568 volts; of car B, 517 volts; of car C, 510 volts. Miscellaneous losses (friction, etc.) of each car are equal to 15 per cent of the input to that car. Determine: (a) the current through each car; (b) the total power input to each car; (c) the copper loss in each car; (d) the miscellaneous losses of each car; (e) the total loss in each car; (f) the horsepower output of each car; (g) the efficiency of each car.

2.16. Show that, if a resistance R' is connected in series with E', the current through R' is

$$ I = \frac{I_0'}{1 + \dfrac{R'}{\Delta}\,(R_1 + R_2)(R_3 + R_4)} $$

in which $\Delta = R_1R_2R_3 + R_2R_3R_4 + R_3R_4R_1 + R_4R_1R_2$ and I_0' is the value of I for $R' = 0$.
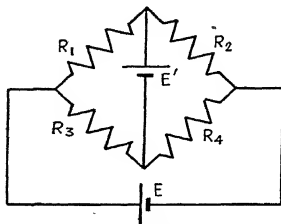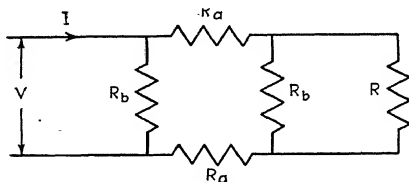
PROB. 2.16.

2.17. Two No. 10 A.W.G. conductors are laid along the ground to feed a 5-kw. motor 200 yd. from the voltage supply. The leakage conductance is 10^{-4} mhos per ft. If the voltage at the motor is 230 volts, calculate the sending end voltage and the efficiency of transmission.

2.18. Calculate the resistances in a three-section T network such that the input and output resistances are 200 ohms and the ratio of output voltage to input voltage is 0.027.

2.19. Show that $\sqrt{R_s R_0} = R_b \sqrt{\dfrac{R_a}{R_a + R_b}}$

in which $R_s = (V/I)_{R=0}$, and $R_0 = (V/I)_R$.

PROB. 2.19.

2.20. A single-strand cable has a resistance of R ohms per unit length. The sheath resistance is negligible. The conductance from wire to sheath is $1/R'$ mhos per unit length. Prove that:

$$R_c = \sqrt{R_0 R_s},$$

where R_c is the resistance measured from one end of the wire to the sheath of a very long piece of the cable; R_0 is the resistance measured from one end of the wire to the sheath of a length a of the cable, when the other end of this wire, distant a from the measuring point, is unconnected; and R_s is the resistance measured from one end of the wire to the sheath of a length a of the cable, when the other end of this wire, distant a from the measuring point, is connected to the sheath.

2.21. Referring to the figure on page 73, find n for the input and output conditions indicated in the diagram.
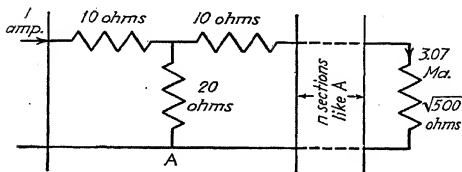
Prob. 2.21.

2.22. A wire of length a has a total resistance Ra. It is surrounded by an imperfect insulating material, which, in turn, is covered by a grounded metal shield of negligible resistance. Throughout the length of the wire the conductance from wire to the grounded shield is

$$\frac{1}{R'} \qquad \text{mhos per unit length}$$

One end of the wire is connected to a terminal of a battery. The other end of the wire and the other terminal of the battery are grounded. Show that the current in the wire at the grounded end is the same as the current that would flow if the insulating material were perfect and the wire had a total resistance of

$$\sqrt{RR'} \sinh a \sqrt{\frac{R}{R'}} \qquad \text{ohms}$$

2.23. A wire of resistance r ohms per meter forms a circle a meters in diameter. A radial wire of the same kind makes sliding contact with the circular wire and rotates with constant angular velocity ω radians per second. Calculate the resistance R between a point A on the circular wire and the end of the radial wire at the center O of the circle as a function of time. If a constant emf E is connected to A and O, what is the average power dissipated in the resistance R? Choose numerical values for E, r, a, and ω, and plot a curve showing how the current varies with time.

2.24. The total resistance of a concentric-conductor cable is 10^{-4} ohm per m. and the leakage resistance of the insulation is 10^{11} ohms per meter. Calculate the currents through a 100-volt battery connected to one end of a 10,000-km. cable if the resistance connected to the other end is (a) zero, (b) infinite, (c) 1,000 ohms.

2.25. How can 320 three-ohm resistors be interconnected so that a current of $\epsilon(2.71828)$ amp. flows through a source of emf of 1 volt connected to two terminals of the network?

SOURCES OF ELECTROMOTIVE FORCE; RESISTANCE OF CONDUCTORS

Having derived methods for calculating the steady currents in d-c circuits, it is convenient now to discuss briefly some of the sources of emfs and a few of the properties of conductors commonly used in such circuits. The physical phenomena underlying these apparently simple devices and the theories devised to correlate them are extraordinarily complicated. The reader is warned that the simple interpretations presented in this chapter are far from complete. Even the modern interpretations of some of these phenomena, to which reference is made in the bibliography (page 102), are ·incomplete and unsatisfactory. However, from the engineering point of view, many complex physical phenomena can be represented accurately enough for *most* practical purposes by relatively simple theories. Thus one of the engineer's most important tasks is to learn to know when the technical approximations that he habitually uses are applicable, and when he must seek methods that are more comprehensive and more accurate than those he commonly uses.

3.01. Electronic Conduction in Metals.—In Chap. I the charge (-1.59×10^{-19} coulomb) and the mass (9.035×10^{-31} kilogram) of the electron were listed and a brief theory of metallic conduction was presented. This theory is based upon the assumption that some of the electrons associated with atoms of the metals are so loosely held under the influence of their nuclei that they _can drift through the volume of the metal to constitute an electric current.* This theory can be usefully expanded into a qualitative interpretation of some of the phenomena that are associated with sources of emf.

Since so many electrons take part in the flow of currents of practical magnitudes (1 microampere is a current of the order of 10^{14} electrons per second), a theory based upon the activity of a single electron is not likely to be useful from the engineering

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* In most cases, one electron per atom takes part in this phenomenon.

point of view.   The situation is analogous to the analysis of the pressure of a gas on the walls of a container.   Certain macroscopic effects such as the change of pressure with temperature can be satisfactorily interpreted in terms of a simple theory that assumes that the pressure is the *average outward force* produced by the impacts of gas molecules on the walls of the container. No attempt to specify the instantaneous positions and velocities of all the molecules is necessary.   Similarly, a theory of electronic conduction whose quantitative formulas involve the average effects of the positions and motions of many electrons is often adequate.

It should be emphasized at the outset that mechanical models of atomic theories are, at best, *aids* to understanding; they are *representative* of a limited group of facts and they *are not* replicas of the things they represent.   For example, it was suggested 25 years ago that electrons whirled about the nuclei of atoms in certain specified orbits.[1]   The author of this theory, Niels Bohr, certainly knew the scope of usefulness of this theory.   In fact, he avoided talking about the activities of these electrons except when they were assumed to "jump" from one specified orbit to another.   Other writers, however, were not so careful, with the result that an atomic model resembling a miniature solar system became popular.   This seemed to imply that the electron was a little ball whirling about and being struck with other balls and that the nucleus was a smaller though heavier ball that was not greatly disturbed by lightweight balls like electrons.   The discovery of the diffraction of electron beams by crystals[2] following the theoretical prediction of certain wavelike properties of electrons knocked the props from the comfortable popular theory that all matter consisted of little spheres in motion.   The quantum theory of radiation,[3] which showed that radiant energy could sometimes usefully be assumed to be a wave and sometimes a stream of (corpuscular) quanta, aided and abetted the popular confusion.   The qualitative theories discussed in this chapter are aids to partial understanding, not comprehensive physical theories.

It is assumed here that a conductor such as a piece of copper wire consists of a fixed framework of copper nuclei.   There is evidence[4] that these nuclei and most of their external electrons form a regular spatial pattern in the form of crystals.   Although

their average positions are assumed to be fixed, they are believed to have associated with them certain kinetic energies, perhaps because they oscillate back and forth about their average positions. These energies increase with temperature.

A few external electrons, perhaps only one per atom of copper, are assumed not to be held firmly to their parent nuclei. Thus these *free* electrons form a cloud (often called an *electron gas*), which may be caused to move through the crystalline lattice of the copper. Under normal conditions (room temperature, no current) the free electrons are assumed to be constantly in motion in irregular (zigzag) paths. They are attracted by positively charged nuclei or positive ions and repelled by negatively charged electrons. Their interactions with other subatomic entities are assumed to be in accord with the law of conservation of energy. Within the metal the forces on them are relatively small, but just outside the surface of the metal these forces increase greatly. The crystal lattice is assumed to produce at the surface a *potential barrier* of such magnitude that the kinetic energies of free electrons at room temperature are insufficient to permit the electrons to escape, *i.e.*, to diffuse outward from the surface. However, if the temperature of the metal is increased by heating it, the kinetic energies of free electrons may become sufficiently high to overcome the surface barrier. They are then ejected outward from the surface of the metal by the process known as *thermionic emission* (described in Chap. XV).

The important conclusions derived from this theory are that in a metal (1) the electron gas is confined to the volume of the metal by an electrical barrier; (2) the electron gas can be caused to drift within the volume of the metal by any force that tends to move an electric charge; and (3) a drift of free electrons through a conductor is accompanied by a transfer of some of the kinetic energy of drift to the relatively fixed components of the metal, *i.e.*, into heat.

3.02. Sources of Electromotive Force.—A source of emf may be described in terms of the theory presented above as a two-terminal device in which energy is so transformed that there is an excess of electrons on one terminal (negative) and a deficiency of electrons on the other terminal (positive). If a network of conductors is connected to the source, the emf may cause the free electrons in these conductors to move, forming an electric

current in each of them. If the currents remain essentially constant after they are initiated, the currents are called *direct* or *continuous currents* (Chap. II). If they vary cyclically with time they are called *alternating currents* (Chaps. V to VII). In this case the electron drift velocity starts at zero, increases to a maximum, decreases to zero, increases in the opposite direction, decreases to zero, and then repeats this cycle. Thus the free electrons execute a complete cycle of oscillation in one cycle, returning at the end of the cycle to the positions (on the average) they occupied at the beginning of the cycle. Finally currents that vary with time in such a manner that they approach a steady state asymptotically are called *transient currents* (Chap. IV). In all cases there is a transformation of energy in the conductor into heat. The rate of transformation into (joule) heat is proportional to the square of the magnitude of the current at a given instant.

The most common sources of either direct or alternating current are *generators*. In general these devices consist of one or more pairs of magnetic poles and one or more insulated conductors; a source of energy such as a steam turbine produces relative motion of the poles and conductors. This procedure gives rise to an *induced emf* in the conductors. This induced emf in turn can cause a current to flow in a circuit external to the generator as well as in the conductors of the generator. This phenomenon is discussed in detail in Chap. XII. It is there shown that when an electric charge moves in the vicinity of a magnet, a mechanical force acts on the charge in a direction perpendicular to its motion and to the direction of the magnetic field produced by the magnet. In terms of the theory outlined above, the electron gas is subjected to a force when the conductor containing it is moved relatively to a magnet. This causes the electron density to increase in one part of the conductor and to decrease in another part. The work (joules) necessary to transfer a unit positive charge (coulomb) from the site of the excess of free electrons to the site of the deficiency of electrons is the difference of electric potential (volts) between these points. This difference of potential is, in the absence of a current through the conductor, equal to the induced emf (volts).

If there is a temperature gradient in a conductor, a small but measurable emf is produced between two points in the conductor

at different temperatures. This and associated phenomena are called the *Thomson effect*. If this emf were always observed to have its positive terminal at the point of higher temperature, it could be assumed that the increased temperature caused a diffusion of the electron gas from points at higher to points at lower temperature, in accord with the theory here assumed. This is true for some metals, such as copper, but the effect is reversed for others, such as iron. Thus the theory fails in this case.

It might be supposed that the surface of contact between two different metals would be the site of measurable and interesting phenomena because the theory here used suggests a difference in the properties of the electron gases in the two metals and in the potential barriers at the surface of contact. Although free electrons cannot overcome the potential barrier between the surface of a metal and a gas or vacuum at room temperatures, the potential barrier at the surface of contact between two metals can apparently be overcome by these electrons. Moreover, since the electron gases of two metals can be assumed to have different properties, it might be expected that more electrons would diffuse from a metal A to a metal B than vice versa, thus producing a *contact emf*. Experiment demonstrates such emfs, although the quantitative expression of the theory[5] of the phenomenon is complicated. Although contact emfs between metals are from a few tenths of a volt to one volt or more, their resultant emf in a closed circuit at constant temperature is zero. Thus if two wires of different materials are connected to form a closed circuit, no current flows if all parts of the closed loop are maintained at the same temperature. If one junction is maintained at a higher temperature than the other a *thermoelectric current* flows because the emfs at the two junctions are no longer equal and opposite. A *thermocouple*, together with suitable electrical measuring instruments, can be used to measure temperature. The thermoelectric, or Seebeck, effect is also used as the basis of current–measuring instruments.

Peltier observed and described an effect (now bearing his name) that is the converse of the effect described above. He showed that if a current flows through a junction from a metal A to a metal B, and, at the other end of the conductor B, from B to A, heat is evolved at one junction and absorbed at the other. If the direction of current flow is reversed, the effects at the two

junctions are reversed. The *Peltier emfs* at the two junctions are oppositely directed in relation to the current flow. If a current of electrons traverses the region of the Peltier emf from its positive to its negative terminal, the emf expends energy. The source of energy is the intrinsic heat of the metals at the junction. As this is expended, the temperature decreases. On the other hand, a current of electrons in the second junction from its negative to its positive terminal delivers energy to the junction, *i.e.*, causes its temperature to rise.

These effects are independent of the joule heating effect (ri^2) of a current in a conductor. Part of the theoretical complication arises from the fact that several effects may occur simultaneously. For example, consider two wires of different materials A and B connected to form a closed loop. If one junction is maintained at a higher temperature than the other, a thermoelectric current flows in the loop. According to the discussion above there are at least three sets of emfs acting in the loop:

1. Two different contact emfs whose difference is the chief cause of the flow of the thermoelectric current.

2. Two different Thomson emfs produced by the temperature gradients along the two wires.

3. Two Peltier emfs causing evolution of heat at one junction and absorption of heat at the other, produced by the current flow through the junctions.

Finally there is drop of potential (ri) along each conductor because of the joule heating effect (ri^2) of the current.

The *photoelectric effect* is a phenomenon assumed to be produced by the transfer of energy from a beam of light to electrons in a metal in such a manner that electrons escape from the metal. This phenomenon and its important engineering applications are discussed in Chap. XV.

Batteries are commonly and conveniently divided into two classes. A *cell* of either class consists of two electrodes immersed in an *electrolyte* (see Sec. 3.03). One or both electrodes or the electrolyte of a *primary cell* may deteriorate with use in such a manner that the cell cannot be restored for further use by electrical means. In the secondary cell, on the other hand, the process of deterioration caused by use is reversible. The cell can be *recharged* by causing a current to flow through the cell in a direction opposite to the emf of the cell.

The theory of the transformation of chemical energy in a battery into electrical energy in a circuit connected to it is complicated and not yet complete or rigorous.[6] The following summary theory of the operation of a *dry cell* is presented to illustrate the phenomena involved in such problems. The negative electrode of a dry cell is a zinc cup. This cup is lined with cardboard, which is saturated with a solution of ammonium chloride and zinc chloride. The positive electrode comprises a carbon rod around which there is a cloth bag containing a pastelike mixture of carbon, manganese dioxide, ammonium chloride, and water. This assembly is suspended along the axis of the zinc cup and the space between the lining of the cup and the cloth bag is filled with a pastelike mixture of ammonium chloride, zinc chloride, gelatin, and water. The top of the zinc cup is sealed, with the positive terminal projecting from the center, by means of an insulating wax. As current is taken from the cell, whose open-circuit emf is about 1.5 volts and whose resistance is a few hundredths of an ohm, the zinc cup is slowly dissolved, the electrolyte undergoes chemical changes, the carbon rod remains unchanged, the emf decreases, and the internal resistance increases. According to one theory, zinc goes into solution to form positive zinc ions, each having a charge equal to that of two protons $(2 \times 1.59 \times 10^{-19}$ coulomb). This process leaves the zinc cup negatively charged. If the cell is not connected to an external circuit, an equilibrium condition is apparently reached such that the tendency for zinc to go into solution in the electrolyte is equal to the tendency for zinc ions to be attracted to the negatively charged zinc cup. What occurs at the positive terminal is more difficult to interpret. Apparently hydrogen ions $(+1.59 \times 10^{-19}$ coulomb) give up their charges to the carbon, become hydrogen gas, and react with the manganese dioxide to form water and some other oxide of manganese. In the absence of manganese dioxide, hydrogen bubbles collect on the carbon rod, thus insulating the rod from the electrolyte. The process is called *polarization* and the manganese dioxide that prevents the formation of an insulating layer of gas on the positive electrode is called a *depolarizer*.

Note that many of the effects discussed in this section are associated with the evolution or absorption of heat. The theories

of thermodynamics are used to correlate the observed phenomena according to modern theories.[7]

3.03. Electrolysis.—Ordinary salt (sodium chloride) in solid form is assumed to comprise electrically neutral molecules each of which consists of one atom of sodium and one atom of chlorine held together in a pattern that depends upon the properties of the external electrons of the two atoms. When salt is dissolved in water it is assumed that the molecules dissociate into a *sodium ion* having a unit positive charge ($+1.59 \times 10^{-19}$ coulomb) and a *chlorine ion* having a unit negative charge (-1.59×10^{-19} coulomb). Such a solution is an electrical conductor. It is assumed that a current through a solution of salt (between platinum electrodes, for example) comprises negative chlorine ions moving toward the positive platinum electrode and positive sodium ions moving toward the negative platinum electrode. When the sodium *ions* arrive at the negative electrode they acquire an electron to become sodium *atoms* and react with the water to form sodium hydroxide and hydrogen that is evolved at this pole. The chlorine *ions*, on the other hand, are assumed to give up an electron at the positive electrode to become chlorine atoms; chlorine is evolved at the positive pole. This is a simple example of *electrolysis*.

A summary theory of electrolysis is presented below. It combines a synopsis of the work of Faraday[8] and the modern ionic theory of electrolytic conduction. It is a far cry from the complicated modern theory of electrochemistry and the extensive experimental data of this subject, concerning which one author[9] has apologized for the difficulty of presenting an adequate survey in a book of 465 pages.

It is assumed that (1) all ions of a given kind are electrically alike, (2) each ion has associated with it a charge equal to an integral multiple of the electronic charge or of its positive counterpart, (3) the integral multiple is in each case equal to the valence, and (4) moving ions constitute a current flow through an electrolyte. According to these assumptions a direct current I passing through several electrolytes connected in series liberates in unit time equal numbers of equi-valent ions, twice as many univalent ions as divalent ions, etc. The assumptions can therefore be tested experimentally. If α represents a substance

liberated by electrolysis, Z the valence of the substance, I the current in amperes, A_α the atomic weight of α, A'_α the mass in kilograms of one ion of α, N_α the number of ions liberated in 1 second, M_α the mass of α liberated in 1 second, and e the electronic charge in coulombs.

(3.01)   kilograms per second

(3.02)   amperes

from which

(3.03) $\dfrac{A'_\alpha I}{Z_\alpha e}$ kilograms per second

If α is an element, A'_α is proportional to the atomic weight; if α represents a chemical radical (for example, SO_4), A'_α is proportional to the sum of the atomic weights of the component parts of α (for example, $A'_s + 4A'_o$ for the sulphate radical). The factor of proportionality is the same for all substances, and it is called the *Avogadro number*. It is the number of atoms in a kilogram-atom, or the number of molecules in a kilogram-molecule. If this number is combined with the electronic charge e, Eq. (3.03) becomes

(3.04) $= K \dfrac{A_\alpha}{Z} I$ kilograms per second

where $K = \dfrac{1}{Le}$, L being the Avogadro number. Experiments show that the mass of a substance liberated by electrolysis per second is proportional to the current through the electrolyte, in accord with Eq. (3.04).

The value of K can be determined experimentally. In a solution of silver nitrate the valence of silver is $Z = 1$; the atomic weight of silver is 107.88; the mass of silver liberated by a current of 1 ampere is 1.118×10^{-6} kilogram per second. Thus $K = 1.0363 \times 10^{-8}$. Since the electronic charge is 1.59×10^{-19} coulomb, the Avogadro number $(1/Ke)$ is 6.06×10^{26} atoms per kilogram-atom. Thus Eq. (3.04) is equivalent to

(3.04a) $= \dfrac{1}{96.5 \times 10^6} \dfrac{A_\alpha}{Z} I$ kilograms per second

The quantity of a substance having a mass in *grams* equal to the

quotient of the atomic weight and the valence is often called the *combining weight*. Note that 96.5×10^3 coulombs liberates the mass in *grams* equal to the combining weight A_a/Z. This quantity of electricity is called the *faraday*.

These relations, Eqs. (3.01 to 3.04a), were not derived theoretically, as described above. The data of carefully controlled experiments led Faraday to two conclusions: (1) the mass of a substance liberated by electrolysis is proportional to the quantity of electricity that passes through the electrolyte, and (2) the mass of a substance liberated by electrolysis is proportional to its combining weight. Thus he studied the phenomena that underlie modern electrolytic processes such as the purification of metals, electroplating, the production of certain chemicals, and the standardization of the measurement of current by means of the silver voltameter.

3.04. Resistance of Conductors.—Experiments show that the resistance of a straight wire of uniform cross section is proportional to its length and inversely proportional to its cross section and depends upon the nature of the material of which it is made and its temperature. The proportionality between the resistance and the length of a conductor, and the inverse proportionality between the resistance and the cross section of a conductor, hold only when (1) the two end surfaces of the conductor, through which current enters and leaves, are parallel, (2) the cross section of the conductor is of the same shape and dimensions throughout, and (3) the emf, if any, within the conductor has the same value in every current path (stream line) through it. Under these conditions the flow of electricity through the conductor is along lines parallel to its axis, each of these lines of flow has the same length, and the current densities at all points in the conductor are equal.

The concepts introduced in Sec. 1.07 are reviewed and developed further in this section.

When any of the three conditions stated is not fulfilled, the flow of electricity through the conductor may not be along parallel lines; current density will in general vary from point to point in the conductor. The lines of flow may be divergent or convergent, either straight or curved, and either of the same or of unequal lengths. The particular shape and distribution of these lines of flow, or *stream lines*, depend upon both the shape of the con-

ductor and upon the location of the terminals at which the current enters and leaves it.

A surface every point of which is at the same electric potential, *i.e.*, between any two points of which no difference of electric potential exists, is called an *electric equipotential surface.* Between any two equipotential surfaces between which a finite difference of potential exists, there must exist an infinite number of equipotential surfaces, each separated from the next by an infinitesimal distance and differing in potential therefrom by an infinitesimal amount.   Since the electric potential at any point
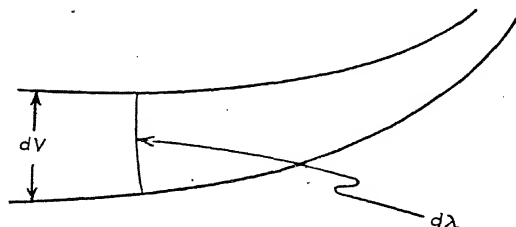
FIG. 3.01.

can have but a single value, no two of these equipotential surfaces can intersect each other.

Imagine two equipotential surfaces that are infinitesimally close together, and let $d\lambda$ be the perpendicular distance between these two surfaces at any point P.   Let dV be the difference of potential between the two surfaces.   Then the quotient $dV/d\lambda$ is called the electric potential gradient at the point from which $d\lambda$ is drawn.

The potential drop along a given distance divided by this distance is equal to the potential drop per unit length; hence, the *magnitude* of the potential gradient at any point may be defined in words as the *drop of potential per unit length at this point*, it being understood that the length is infinitesimally small and is perpendicular to the equipotential surface through this point. These concepts are discussed in detail in Chap. X.

It is assumed in the following development that the flow of electricity at any point in a body is always perpendicular to the equipotential surface through that point.   A line drawn tangent at each point to the direction of the flow of electricity

past that point is called a *line of flow* of electricity, or, briefly, a *stream line* of the electric current. The stream lines of an electric current and the electric equipotential surfaces are therefore always mutually perpendicular. In Fig. 3.02 are shown, in two dimensions only, the electric equipotential surfaces (dotted) and the stream lines in a carbon block, when the current is led into and out of the block through two small cylindrical terminals represented by the black dots in the figure. The equipotential surfaces were determined experimentally by
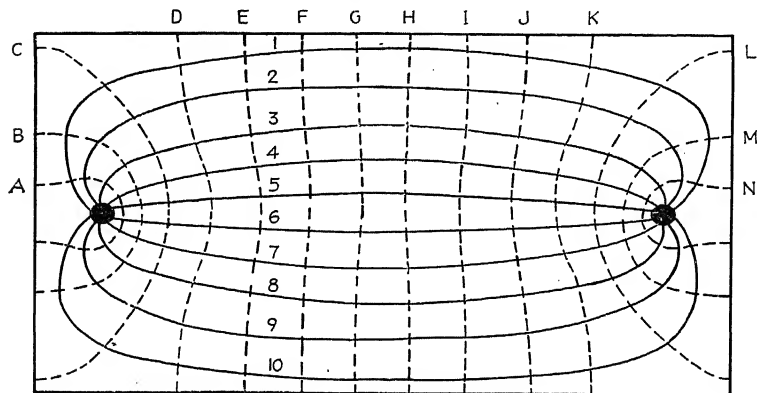
FIG. 3.02.

means cf a potentiometer.[10] In Fig. 3.02, the equipotential surfaces are represented by the broken lines *A*, *B*, *C*, . . . *N*; the stream lines are represented by the solid lines 1, 2, 3, . . . 10.

The *density* of the current at any point in a conductor is defined as the current *per unit area* that flows through a small area at this point taken normal to the direction of the stream line through this point. Such an area is part of an equipotential surface. Let ΔS be such an area at any point P in a conductor, and let ΔI be the current through this area, then the magnitude of the current density at the point P is

$$(3.05) \qquad J = \frac{\Delta I}{\Delta S} \quad \text{amperes per square meter}$$

and the *direction* of the current density is the direction of current flow. Thus current density is a vector quantity.

The distribution of the current in a conductor may be conveniently represented graphically, by drawing in this conductor such a number of stream lines that their number per unit area normal to their direction is equal, for each such area in the conductor, to the current density at this area. The stream lines shown in Fig. 3.02 are so drawn.

Any region in which a potential gradient exists may be thought of as divided into cells (like a honeycomb) whose ends are formed by equipotential surfaces and whose sides are therefore parallel to the direction of the electric current in these cells. These cells may be thought of as being as small as desired, and in the limit each cell may be considered a right cylinder with its axis perpendicular to the equipotential surfaces that form its ends. Each of these small cells then satisfies the three conditions listed in the first paragraph of this section. The resistance R_c of a cell is therefore proportional to its length $\Delta\lambda$ and inversely proportional to its cross section ΔS. The resistance is

$$(3.06) \qquad R_c = \rho \, \frac{\Delta\lambda}{\Delta S} \qquad \text{ohms}$$

where ρ is the factor of proportionality. From Eq. (3.05) the current ΔI through this cell is

$$(3.07) \qquad \Delta I = J \, \Delta S$$

Therefore the drop of potential ΔV_r through the cell attributed to its resistance is

$$(3.08) \qquad \Delta V_r = \qquad = \rho J \, \Delta\lambda$$

From this it follows that, if the length and cross section of the cell approach zero as a limit

$$(3.09) \qquad \frac{dV_r}{d\lambda} \qquad \text{volts per meter}$$

The quantity $dV_r/d\lambda$ is the drop in electric potential per unit length in the direction in which the current flows. Note that, from the definition of potential drop given in Sec. 1.06, $dV_r/d\lambda$ has the dimensions of *force per unit charge*. This quantity is called the *electric intensity* $\mathbf{E}$ at the point P. From Eq. (3.09), which represents the relation between the *magnitudes* of the potential

gradient and the current density, the corresponding vector relation is

$$(3.10) \qquad\qquad \mathbf{E} = \rho J \qquad\qquad \text{volts per meter}$$

The direction of the electric intensity at any point in a conductor is defined as the direction of the electric current at this point, *i.e.*, the electric intensity at a given point in a conductor has the direction of the stream line of the current through that point.

There are a number of useful configurations of conductors for which: (1) the current density J is a function only of the distance from an equipotential surface along any current stream line [*i.e.*, $J = f(\lambda)$], (2) the factor ρ is constant throughout the conductor, and (3) at every equipotential surface J is constant over the surface.   If these conditions are satisfied, the volume between two equipotential surfaces in the conductor may be divided into a number of equal volumes each having surfaces ΔS_A and ΔS_B on the equipotential surfaces and other bounding surfaces everywhere parallel to the current stream lines.   The resistance of one of these volumes is

$$(3.11) \qquad\qquad \iota_{\Delta AB} = \frac{}{\lambda \cdot \mathbf{T}} = \frac{{}'J_\lambda \, d\lambda}{\Delta I} \qquad\qquad \text{ohms}$$

where J_λ is a function only of λ, and ΔI is that part of the total current between A and B that flows in this volume.   But since the resistances of all volumes are equal, their parallel resistances, *i.e.*, the net resistance of the conductor between A and B is equal to $R_{\Delta AB}$ divided by the number of volumes.   This number is equal to $S_A/\Delta S_A = S_B/\Delta S_B$.   The net resistance R_{AB} between A and B is therefore

$$(3.12) \qquad\qquad R_{AB} = \frac{{}'\int_A^B J_\lambda \, d\lambda}{\Delta I} \frac{\Delta S_A}{S_A}$$

But from the conditions listed above

$$\frac{\Delta I}{\Delta S} S_A = I, \text{ the total current flowing between } A \text{ and } B$$

Therefore

$$(3.13) \qquad\qquad\qquad J_\lambda \, d\lambda \qquad\qquad\qquad \text{ohms}$$

Thus the assumptions upon which this development is based may be tested in a few special cases (for example, see Sec. 3.07).

In most engineering problems the current flows in straight cylindrical conductors of which the end surfaces, perpendicular to the axis of the conductor, are equipotential surfaces. In this case $J_\lambda = I/S$ is constant between the ends and Eq. (3.13) becomes $R_{AB} = \rho \dfrac{\lambda}{S}$ ohms, Eq. (3.14). In other words, in this special case, the equation (3.13) reduces to the mathematical formulation of the experimental facts listed in the first paragraph of this section.

3.05. Direct-current Conductance; Resistivity and Conductivity.—The reciprocal of the d-c resistance of a conductor is defined as the *direct-current conductance* of this conductor, *i.e.*, the d-c conductance corresponding to a d-c resistance R is

$$(3.15) \qquad\qquad\qquad \frac{1}{R} \qquad\qquad\qquad \text{mhos}$$

From Ohm's law the potential drop produced by a current I in a conductor of resistance R, when there is no emf in this conductor, is $V = RI$. The current established by a potential difference V impressed across the terminals of a conductor of conductance G, *when there is no emf in this conductor*, is

$$(3.16) \qquad\qquad\qquad = GV \qquad\qquad\qquad \text{amperes}$$

Again, from Joule's law the power dissipated as heat in a conductor of resistance R when a current I is established in it is $P_h = RI^2$. The power dissipated as heat in a conductor *in which there is no emf*, when a potential difference V is impressed across its terminals is

$$(3.17) \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{watts}$$

The unit of conductance has the same dimensions as the reciprocal of the unit of resistance. This has led to the adoption of the term "mho" as the name for the practical unit of conductance, the word "mho" being the word "ohm" written backwards. A conductance of 0.000001 mho is called a micromho. Note that a conductance of 1 *micromho* is equivalent to a resistance of 1 *megohm*.

The d-c resistance R of a straight wire of uniform cross section *at constant temperature* is proportional to its length and inversely proportional to its cross section if each of the end surfaces between which the resistance is measured is an equipotential surface perpendicular to the axis of the wire and if the emf's (if any) in the wire are parallel to its axis and equal for every possible path parallel to the axis.   Under these conditions the resistance is

$$(3.14) \qquad\qquad R = \rho \frac{\lambda}{S} \qquad\qquad \text{ohms}$$

The factor ρ is called the *resistivity* or *specific resistance*.   The resistivity of a conductor depends (1) upon the material of the conductor and (2) upon its temperature.   The numerical value of the resistivity, *viz.*, the numerical value of the factor ρ in Eq. (3.14) also depends upon the units in which the resistance R, the length λ, and the cross section S are expressed.

From Eq. (3.14) it follows that the resistivity of a conductor is numerically equal to the resistance of a wire, or bar, which has unit length and unit cross-sectional area, if the stream lines of the current through this bar are parallel to its axis, and the end surfaces of the bar are parallel equipotential surfaces.

In particular, when the meter is used as the unit of length, the square meter as the unit of area, and the ohm as the unit of resistance, the resistivity is equal to the resistances in ohms of a cube of the material each edge of which is 1 meter long.   The unit is the "ohm-square meter per meter" or, as it is usually called, the *ohm-meter*.

The conductance of a wire, under the conditions stated above, is, according to Eqs. (3.10) and (3.09)

$$(3.18) \qquad\qquad G = \frac{1}{\rho} \frac{S}{\lambda} \qquad\qquad \text{mhos}$$

It has been found convenient to write this expression in the form

$$(3.19) \qquad\qquad G \qquad \gamma \frac{S}{\lambda} \qquad\qquad \text{mhos}$$

where

(3.20)

The factor γ is called the *conductivity* or *specific conductance*.

The conductivity, like the resistivity, depends upon the material forming the wire and the temperature. Its numerical value depends upon the units in which G, λ, and S are expressed.

3.06. The Temperature Coefficient of Electric Resistance.— If the resistance of conductor is measured at a number of temperatures throughout a given range, the resistance R at any temperature $T°$ C. within the range may be expressed in the form:

$$(3.21) \qquad R = R_1 \sum_{n=0}^{n=\infty} \alpha_{n,1}(T - T_1)^n$$

where R_1 is the resistance at some arbitrarily chosen temperature T_1 within the measured range and the coefficients $\alpha_{n,1}$ are determined from the measured resistances. These coefficients for $n > 1$ are found to be *practically* of negligible magnitude for most metallic conductors for temperatures $-50°$ C. $\leq T \leq 200°$ C. For most engineering work, therefore, the resistance of a conductor at any temperature T within this limited range may be calculated from the approximation of Eq. (3.21)

$$(3.22) \qquad R = R_1[1 + \alpha_{1,1}(T - T_1)]$$

The factor* $\alpha_{1,1}$ is called the *temperature coefficient of resistance of the conductor at the temperature T_1*. The value of $\alpha_{1,1}$ depends upon (1) the conductor material and (2) the value of the temperature T_1 and the units in which the resistances and temperatures are expressed. Since none of the coefficients except $\alpha_{1,1}$ is used below, the first subscript may be dropped. The centigrade temperature scale is commonly used, and the temperature T_1 is usefully chosen as $T_1 = 20$ so that

$$(3.23) \qquad R = \qquad + \alpha_{20}(T - 20)] \qquad \text{ohms}$$

where R is the resistance at a temperature of $T°$ C., R_{20} is the resistance at 20° C., and α_{20} is the temperature coefficient of resistance of the conductor at 20° C.

To obtain α_1 when T_1 and α_{20} are known, eliminate R, R_{20}, R_1 from Eqs. (3.22) and (3.23)

$$(3.24) \qquad \frac{\alpha_{20}}{\cdot 20)} \qquad \text{ohm per ohm per degree centigrade}$$

* Note that the α's are assigned two subscripts, the first to designate n, the summation variable in Eq. (3.21), the second to designate the temperature for which the coefficient has been obtained.

3.07. Insulation Resistance of a Single Conductor Cable.—In Sec. 3.04 the following expression was derived for calculating the resistances of certain symmetric configurations of conducting material:

$$J_x \, dx \qquad\qquad \text{ohms}$$

where the distance x is measured in the direction of flow of current. As an example of the use of this relation, the insulation
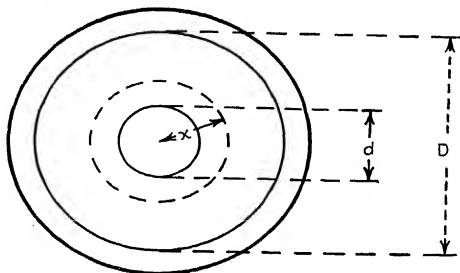
FIG. 3.03.

resistance of a single-conductor cable, shown in cross section in Fig. 3.03, is calculated below. The insulation resistance is conveniently expressed in terms of the diameter d of the conducting core, the external diameter D of the insulation around it, the resistivity ρ of this insulation, and the length Λ of the cable. The insulation of such a cable is frequently covered with a lead sheath as shown. When such a sheath is not provided, the insulation resistance is tested by immersing the cable in a conducting solution, which can then be considered as forming a conducting sheath around it.

Let V be difference of potential established between the core and the sheath that surrounds the insulation, and let I_1 be the current, in amperes, that flows through the insulation from the core to the sheath. This current is usually referred to as the *leakage* current, and is not to be confused with the main current that flows through the core to the load to which the cable may be connected in actual use.

Because of the high conductivity of the core, the drop of potential along it is usually negligibly small in comparison with

the drop of potential through the insulation.   Therefore the surface of the core may be considered to be an equipotential surface.   In the sheath there is no other current than the leakage current, and, because of the relatively high conductivity of the sheath, the external surface of the insulation may likewise be considered an equipotential surface.

The stream lines of the leakage current through the insulation must therefore be perpendicular both to the surface of the core and to the external surface of the insulation, and when the insulation is of uniform thickness, must, from symmetry, be equally spaced radial lines.   Hence the *magnitude* of the current density at any point in the insulation at a distance x from the center of the wire is equal to the total leakage current I_1, divided by the area of the cylindrical surface that passes through this point and that is concentric with the wire

(3.25) $J =$ amperes per square meter

Therefore, from Eq. (3.11)

(3.26)   ohms

Note that the insulation resistance varies inversely as the length. This is evidently true when it is remembered that each elementary length of the insulation of the cable (measured along its axis) is in parallel with all the other elementary lengths.

In Eq. (3.26) the length Λ must be expressed in meters when ρ is expressed in ohm-meters, in order that R be in ohms.   The two diameters D and d may be expressed in any unit of length, provided both are expressed in the *same* unit, for it is their ratio only that enters into the formula.

3.08. Percentage Conductivity; the American Wire Gage.— The international volt, ampere, and ohm are standards universally used in engineering practice.   No such international standard has been accepted for designating the *sizes* of wire used for the construction of electrical apparatus and the distribution of electric power.   The student should learn to use an electrical engineering handbook for most information of this kind.   The American Wire Gage system, which is generally used by electrical engineers *in*

this country, is a remarkable example of: (1) the arbitrary formulation of a set of practical standards without regard to their relation to international scientific standards and (2) the impossibility of changing the practical standards after they are used for some years.   It is for these reasons described briefly below.

It is customary in English-speaking countries to specify the length of a wire in feet and the cross section in *circular mils.*   By a circular mil is meant the area of a circle that has a diameter of 0.001 inch, *i.e.,* a diameter of 1 *mil.*   Since the area of a circle varies as the square of its diameter in mils, this unit of area is convenient for expressing the area of the cross section of a circular wire, for its use eliminates the factor π.   For example, a circular wire having a diameter of 0.5 inch has an area of $(500)^2 = 250,000$ circular mils.

The standard conductivity for copper now in general use is the so-called *Annealed Copper Standard* adopted by the International Electrotechnical Commission in 1914.   This standard is defined as follows:

1. At a temperature of 20° C., the resistance of a wire of standard annealed copper 1 meter in length and of a uniform section of 1 square millimeter is 1/58 ohm = 0.017241 · · · ohm.

2. At a temperature of 20° C., the density of standard annealed copper is 8.89 grams per cubic centimeter.

3. As a consequence, it follows from (1) and (2) that, at a temperature of 20° C. the resistance of a wire of standard annealed copper of uniform section, 1 meter in length and weighing 1 gram is $(1/58) \times 8.89 = 0.15328$ ohm.

By the percentage *conductivity* of a conductor is meant 100 times the ratio of its conductivity to the conductivity corresponding to the Annealed Copper Standard.   Therefore, the lower the percentage conductivity, the higher is the resistivity of a conductor.

The Bureau of Standards recommends that whenever the conductivity of a sample is expressed as a percentage, the measured resistivity or conductivity be corrected to reduce it to the value it would have at 20° C.

It is of interest to note that annealed copper, when very pure, may have a conductivity greater than 100 per cent.   Ordinary annealed copper wire usually has a conductivity between 98 and 100 per cent, and hard-drawn copper wire a conductivity between

96 and 98 per cent.   Commercial aluminum wire has a conductivity of about 62 per cent.

In this country, wires for electrical purposes, when less than $\frac{1}{2}$ inch in diameter, are usually specified in terms of a wire gauge introduced by the Brown & Sharpe Manufacturing Company. This gauge is now known as the American Wire Gage and is abbreviated A.W.G.   The older abbreviation B. & S. is also used to designate this gauge.   Wires $\frac{1}{2}$ inch in diameter, or larger, are usually specified in terms of their cross section in circular mils.

The diameter of the wires corresponding to successive A.W.G. gauge numbers are so chosen that they differ by a constant percentage.   A solid wire 460 mils in diameter is called a *No. 0000 wire* and a wire 5 mils in diameter is called a *No. 36 wire*. The next smaller size to a No. 0000 wire is No. 000, the next smaller size No. 00, the next No. 0, the next No. 1, and so on up to No. 36.   The ratio of the diameters of No. 0000 and No. 36 is $460/5 = 92$, and the ratio of the diameters of successive sizes is constant; this constant is therefore equal to the thirty-ninth root of 92.*   The thirty-ninth root of 92 is approximately equal to the sixth root of 2; hence the following *approximate* relations (since the cross section varies as the square of the diameter and the cube root of 2 is approximately 1.26):

1. The ratio of the cross sections of wires of successive sizes of the A.W.G. is equal to 1.26, the larger number on the gauge scale corresponding to the smaller cross section.   This same relation holds for the weights per unit length of successive sizes for a given length.

2. The ratio of the resistances of wires of successive sizes of the A.W.G. is equal to 1.26, the larger number on the gauge scale corresponding to the larger resistance.

3. An increase of 3 in the gauge number halves the cross section and weight per unit length and doubles the resistance.

4. An increase of 10 in the gauge number divides the cross section and weight by 10 and multiplies the resistance by 10.

The cross section of a No. 10 wire is approximately 10,000 circular mils, its resistance is approximately 1 ohm per 1,000 feet at 20° C., and its weight is approximately 31.5 pounds per 1,000 feet.   From the above relations the resistance, cross

* Can someone suggest a more perplexing basis for a wire gauge?

section, and weight of any size of wire may be calculated approximately with but little effort.   The resistance of a No. 10 aluminum wire is approximately 1.6 ohms per 1,000 feet at 20° C. and its weight approximately 9.5 pounds per 1,000 feet.

The above relations are for solid wire.   The gauge number of a stranded wire corresponds to the cross section of the metal in it, and not to its over-all diameter.   The diameter of a concentric strand is approximately 15 per cent greater than that of a solid wire of the same number; its weight and resistance are from 1 to 2 per cent greater, depending upon the number of twists per unit length and the number of wires in the strand.

Complete wire tables, giving diameters, cross sections, weights (usually pounds per 1,000 feet or per mile), and resistances (ohms per 1,000 feet or per mile), will be found in any electrical engineers' handbook.[11]   The student should familiarize himself with these tables, for he will have frequent occasion to refer to them.

3.09. Numerical Data Related to the Material Discussed in Chapter III.—It is the purpose of this section to present a few experimental data, taken from the International Critical Tables and the Handbook of Chemistry and Physics.   For more extensive data and for a description of methods of measurements the reader is referred to these sources and to their bibliographies.

Although the subject of thermionics is discussed in Chap. XV, the energy required for electrons to overcome the potential barrier at the surface of a substance (Sec. 3.01) is given below in Table 3.01 for several substances; these data were obtained from thermionic experiments.   The data of Table 3.01 are given in terms of the quantities in the equation

$$(3.27) \qquad\qquad J = A T^2 \varepsilon^{-\overline{V_t}} \quad \text{amperes per square meter}$$

J is the time rate of ejection of electrons from unit surface of the thermionic emitter at temperature $T°$ absolute (Kelvin);

$$V_t = \frac{k}{e} T = 8.62 \times 10^{-5} T \qquad \text{volts}$$

(k is Boltzmann's constant $= 1.371 \times 10^{-23}$ joule per degree); V_w is the work function expressed in volts, $i.e.,$ $e V_w$ is the average energy required by an electron to surmount the potential barrier at the surface; the coefficient A and the work function V_w are

listed in Table 3.01 for several substances. Table 3.02 shows observed thermionic current densities for tungsten; they are

TABLE 3.01.—CONSTANTS OF THERMIONIC EMISSION

Thermionic emitter	Temperature, °abs.	A, amp. per m^2 per deg.^{2}	V_w, volts
Carbon....................	2000	6.02×10^5	4.00
Molybdenum.............	2000	6.02×10^5	4.44
Platinum.................	1600	6.02×10^5	5.08
Barium oxide on platinum..	1200	2.88×10^6	1.68
Thorium oxide on platinum	2000	5.7×10^3	3.18
Tungsten.................	2000	6.02×10^5	4.52
Thoriated tungsten........	1600	3×10^5	2.63

TABLE 3.02.- –THERMIONIC EMISSION FROM TUNGSTEN

Temperature °abs.	Emission, J amp. per m.^{2}
1000...........	..... 1.07×10^{-11}
1500...........	..... 9.14×10^{-4}
2000...........	 10
2500...................................	2.98×10^3

TABLE 3.03.—CONTACT POTENTIAL DIFFERENCES
(Approximately 20° C.)

Substances		Potential of A–potential of B, volts
A	B	
Aluminum	Platinum	+1.20
Bismuth	Platinum	+0.35
Copper	Platinum	+0.13
Magnesium	Platinum	+1.05
Potassium	Platinum	+2.8
Tin	Platinum	+0.62
Zinc	Platinum	+0.90
Copper oxide	Sodium	−2.52
Aluminum	Brass	+0.670
Copper	Brass	−0.154
Lead	Brass	+0.396
Platinum	Brass	−0.354
Tin	Brass	+0.318
Zinc	Brass	−0.608

approximately in accord with Eq. (3.27) and the data of Table 3.01 for tungsten. Note that measurements of the constants A

are difficult to perform accurately, because the results of such measurements depend upon the state of the surface of the thermionic emitter; for example, the presence of small amounts of certain gases in the evacuated space surrounding the emitter causes a reduction in the measured value of A.[14]

Table 3.03 is a list of contact potential differences between pairs of substances A and B at room temperatures.

Table 3.04 shows a few values of the emfs acting around each of two commonly used thermocouple circuits; the cold junction temperature is $0°$ C.; the hot junction temperature is listed in the table.   These are average values; each thermocouple is calibrated by measuring *its* emf at each of three or four temperatures.   If the differences between these calibrating measurements and the average values of the table are constant within the limits of permissible error, the constant difference can be applied to the tabular values as a correction for the particular couple.

TABLE 3.04.—ELECTROMOTIVE FORCES OF THERMOCOUPLES

Couple: Platinum–platinum (90 per cent) rhodium (10 per cent)		Couple: Copper–constantan (copper 60 per cent, nickel 40 per cent)	
°C.	Millivolts	°C.	Millivolts
100	0.64	−200	−5.54
200	1.43	−100	−3.35
400	3.24	0	0
800	7.33	100	4.28
1600	16.75	200	9.29
		300	14.86

The student should design a convenient portable device for using a thermocouple to measure temperature and then compare his design with that used by manufacturers of such equipment.

The rate at which heat is evolved when a current flows through a junction of two metals (from A to B) is approximately proportional over a limited range of temperatures to (1) the current I, (2) the absolute temperature T, and (3) a factor $_AQ_B$ called the thermoelectric "power" (units-volts per degree).   The relation so defined describes the Peltier effect; *Joule heating occurs in addition to this effect.*   The Peltier effect is reversible, *i.e.*, $_AQ_B = -_BQ_A$ so that heat may be evolved or absorbed.   Appar-

ently the Joule heating effect is irreversible. The thermoelectric power $_AQ_B$ is *not* a constant; it is assumed to be a linear or quadratic function of temperature

(3.28) $_AQ_B \equiv a + bt + ct^2$ microvolts per degree ($t = {}^\circ C.$),

linear if a and b are measured and quadratic if a, b, c are measured. Table 3.05 lists values of a, b or a, b, c for the two couples described in Table 3.04.

TABLE 3.05.—DATA ON THERMOELECTRIC POWER AND PELTIER EFFECT

Junction	a, μv per deg.	b, μv per deg.^{2}	c, μv per deg.^{3}
Pt–Pt (90 per cent) Rh (10 per cent) (0 < t < 1600°C.)	+ 7.013	+0.640 × 10^{-2}	+0.1932 × 10^{-5}
Cu (58.1 per cent) Ni (41.9 per cent)-Cu (0 < t < 100°C.)	−35.10	−7.74 × 10^{-2}	

The Thomson effect is described quantitatively as follows:

(3.29) $$P_T = \int_{t_1}^{t_2} \sigma I \, dt \qquad \text{microwatts absorbed}$$

I is the current in amperes flowing *from* a point at temperature t_2 °C. *to* a point at temperature t_1 °C.; σ is the *Thomson coefficient* (microvolts per degree):

(3.30) $\sigma = \alpha + \beta t + \gamma t^2$ microvolts per degree

This effect is superposed on the Joule heating effect. Values of α, β, γ, are given in Table 3.06 for three metals.

TABLE 3.06.—THOMSON EFFECT

Material	α, μv per deg.	β, μv per deg.^{2}	γ, μv per deg.^{2}
Copper (−60° < t° < 127°C.)	− 1.42	−0.74 × 10^{-2}	
Platinum (−72° < t° < 128°C.)	+ 9.10	−0.475 × 10^{-2}	+ 4.75 × 10^{-5}
Constantan (+87° < t° < 481°C.)	+20.00	+2.554 × 10^{-2}	−10.05 × 10^{-5}

Table 3.07 contains data of two standard cells, the dry cell, and two storage cells.

TABLE 3.07.—DIRECT-CURRENT SOURCES

Name of cell	Negative pole	Electrolyte	Positive pole	Depolarizer	Emf, volts	Internal resistance, ohms
Weston normal..	Cadmium amalgam	Saturated $CdSO_4$	Mercury	Paste of Hg_2SO_4 and $CdSO_4$.	1.0183 at 20°C.	20–50
Clark standard..	Zinc amalgam	Saturated $ZnSO_4$	Mercury	Paste of Hg_2SO_4 and $ZnSO_4$	1.4328 at 15°C.	20–50
Dry cell.........	Zinc	NH_4Cl	Carbon	MnO_2	1.53	0.05 –0.1
Lead storage.....	Lead	H_2SO_4 Sp. Gr. 1.2	PbO_2	....	2.2	0.004–0.02
Edison storage...	Iron	KOH 20 per cent	An oxide of nickel	....	1.1	0.003–0.012

Table 3.08 lists the electrochemical equivalents for several metals.

TABLE 3.08.—ELECTROCHEMICAL EQUIVALENTS

Substance	Valence	Mg. per coulomb
Ag	1	1.118
Al	3	0.093
Au	3	0.681
Au	1	2.043
Cu	2	0.329
Cu	1	0.658
Hg	2	1.039
Hg	1	2.079
Na	1	0.238
Pb	4	0.537
S	6	0.055
S	4	0.083
S	2	0.166
W	6	0.318

The resistivities of several conductors and their temperature coefficients of resistivity are listed in Table 3.09. The volume resistivities of several insulating materials are listed in Table 3.10. Note that the surface resistivities of dielectrics usually decrease sharply as the humidity increases and that the volume

TABLE 3.09.—RESISTIVITY AND TEMPERATURE COEFFICIENT OF RESISTIVITY OF CONDUCTORS

Conductor	Resistivity (ohm-m)(10^8)		Temperature coefficient	
Aluminum........	2.828	20	0.0039	18
Brass............	6.4–8.4	0	0.001–0.002	15
Carbon..........	3500	0	−0.0005	
Constantan.......	44.1	0	0.000008	12
Cu 60 per cent				
Ni 40 per cent				
Copper..........	1.7241	20	0.00393	20
Gold............	2.44	20	0.0034	20
Mercury.........	95.783	20	0.00089	20
Platinum.........	10	20	0.003	20
Pt (90 per cent } Rh (10 per cent }	21.14	0	0.0013	0
Silver............	1.629	18	0.0038	20
Silver...........	0.009	−258.6		
Lead............	22	20	0.0043	18
Lead............	0.59	−252.9		
*Lead...........	Approx. 10^{-11}	−270		
Tungsten........	5.51	20	0.0045	18
Tungsten........	59.4	1727	0.0089	1,000

* Example of superconductivity.

TABLE 3.10.—RESISTIVITY OF INSULATING MATERIALS (AT 15° TO 22° C.)
Notes: (1) Volume resistivity (listed below) varies with temperature; (2) surface resistivity (not listed below) varies with humidity.[13]

Material	Approximate Volume Resistivity, ohm-m
Amber.............................	5×10^{14}
Bakelite (pure)......................	2×10^{14}
Bakelite (paper filler)................	2×10^9
Ceresin............................	5×10^{16}
Plate glass.........................	2×10^{11}
Mica..............................	$4 \times 10^{11} - 9 \times 10^{13}$
Porcelain..........................	3×10^{12}
Fused quartz.......................	5×10^{16}
Sulphur............................	1×10^{15}
Paraffined wood....................	$5 \times 10^9 - 4 \times 10^{11}$
Petroleum..........................	2×10^{14}
Ethyl alcohol.......................	3000
Distilled water.....................	5000

resistivities of insulators change markedly as the temperature is changed.

TABLE 3.11.—DIMENSIONS AND RESISTANCE OF COPPER WIRE

Gage No., A.W.G.	Diameter, mils at 20° C.	Cross section at 20° C., cir. mils	Ohms per 1,000 ft. at 20° C.	Ft. per lb.
Concentric lay cable	61–99.2	600,000	0.0180	0.65
Concentric lay cable	37–104.0	400,000	0.0270	0.806
Concentric lay cable	27–82.2	250,000	0.0431	1.297
0000	460.0	211,600	0.0490	1.561
0	324.9	105,500	0.0983	3.13
1	289.3	83,690	0.1239	3.947
2	257.6	66,370	0.1563	4.977
3	299.4	52,640	0.1970	6.276
4	204.3	41,740	0.2485	7.914
5	181.9	33,100	0.3133	9.980
6	162.0	26,250	0.3951	12.58
7	144.3	20,820	0.4982	15.87
8	128.5	16,510	0.6282	20.01
9	114.4	13,090	0.7921	25.23
10	101.9	10 330	0.9989	31.82
11	90.74	8,234	1.260	40.12
12	80.81	6,530	1.588	50.59
13	71.96	5,178	2.003	63.80
14	64.08	4,107	2.525	80.44
15	57.07	3,257	3.184	101.4
16	50.82	2,583	4.016	127.9
17	45.26	2,048	5.064	161.3
18	40.30	1,624	6.385	203.4
19	35.89	1,288	8.051	256.5
20	31.96	1,022	10.15	323.4
21	28.46	810.1	12.80	407.8
22	25.35	642.4	16.14	514.2
23	22.57	509.5	20.36	648.4
24	20.10	404.0	25.67	817.7
25	17.90	320.4	32.37	1031
26	15.94	254.1	40.81	1300
27	14.20	201.5	51.47	1639
28	12.64	159.8	64.90	2067
29	11.26	126.7	81.83	2607
30	10.03	100.5	103.2	3287
31	8.928	79.70	130.1	4145
32	7.950	63.21	164.1	5227
33	7.080	50.13	206.9	6591
34	6.305	39.75	260.9	8310
35	5.615	31.52	329.0	10480
36	5.000	25.00	414.8	13210
37	4.453	19.83	523.1	16660
38	3.965	15.72	659.6	21010
39	3.531	12.47	831.8	26500
40	3.145	9.888	1049.	33410

The American Wire Gage tables are based on the mil (0.001 inch) and the foot as units of length and the circular mil (area

of circle 0.001 inch in diameter) as the unit of area. Since these tables are so widely used in practice the units have not been changed to those of the mks system.

PHYSICAL CONSTANTS MENTIONED IN CHAP. III

Avogadro number...... 6.064×10^{26} molecules per kg.-molecule

Loschmidt number...... 2.705×10^{25} molecules per cu. m. for ideal gas at 0° C.

The Faraday........... 9.649×10^7 coulombs per kg. equivalent

Boltzmann constant.... 1.371×10^{-23} joule per deg.

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9. See 6 above, Preface.
10. See 5 above, pp. 129–134.
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Problems

3.01. In a silver-plating device, 20 g. of silver are deposited in 1 hr. Calculate the quantity of electricity that traverses the device and the average current through it.

3.02. A column of a 15 per cent solution of copper sulphate is 1 m. long and its cross section is 1 mm.^{2} The resistance of this column (at 20° C.) is

260,000 ohms. Calculate the current that would flow in such a solution (at 20° C.) between two plane electrodes 25 by 25 cm., separated by a distance of 2 cm., if the difference of potential between the surfaces of the conducting fluid is 1.8 volts.

3.03. How many atoms of silver are deposited in 1 sec. by a constant current of 1 ma. flowing through a silver voltameter? What mass of silver is deposited in 30 min.?

3.04. How long will it take to fill a balloon 1 m. in diameter, at a pressure of 1.2 atmospheres and a temperature of 20° C., with hydrogen by electrolysis at a current of 10 amp.?

3.05. A copper-constantan thermocouple is to be used to measure the temperature in underground ducts in which power cables are placed. Prepare a table showing the net voltage output when the hot junction is at temperatures from 40° C. to 80° C. (in steps of 5° C.) and the cold junction is at temperatures from 20° C. to 30° C. (in steps of 2° C.).

3.06. Calculate the electrochemical equivalent of oxygen and the time required to fill the balloon described in Problem 3.04.

3.07. Derive the equation for the resistance between the parallel faces of the frustum of a cone in terms of the original height h of the cone, the radius r of its base, the height a of the frustum, and the resistivity ρ of the material of which the cone is made.

3.08. The inner radius of a semicircular conductor is r meters; its thickness in the direction of r is a meters and the other dimension of its cross section is b meters. If the resistivity of the material is ρ ohm-meters, find the resistance between the two plane end faces.

3.09. A No. 18 A.W.G. steel wire forms the core of a conductor whose annular coating is aluminum having an outside diameter equal to that of a No. 6 A.W.G. wire. Calculate the percentage of the total current parallel to the axis of the conductor that is carried by the steel and the percentage carried by the aluminum.

3.10. Two iron wires, each having a resistance of 10 ohms at 0° C., are connected in parallel to a constant emf that produces a total current of 1 amp. when the conductors are at a temperature of 0° C. If one wire is then brought to a temperature of 16° C. and the other to −16° C., while the emf remains constant, calculate the current in each wire.

3.11. B_1 and B_2 are storage cells exactly alike; G is a sensitive galvanometer; P is a coil of standard annealed copper wire, the resistance of which is 1,000 ohms at 20° C.; Q is a coil of nichrome wire of 1,010 ohms resistance at 20° C. The two coils are immersed in a liquid at room temperature (20° C.) and the bath is heated by an external source. What is the temperature of the bath at which the galvanometer indicates zero current?
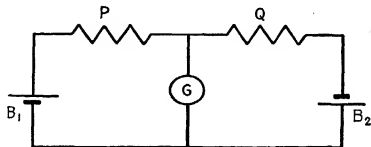
PROB. 3.11.

3.12. An ellipsoid of revolution is formed by rotating the ellipse

about the x axis. If the ellipsoid is made of material of resistivity ρ ohm cm.²/cm.: (*a*) Find the resistance of the solid between two parallel planes, perpendicular to the x axis, passing through the foci of the ellipse. (*b*) If two parallel planes perpendicular to the y axis and equidistant from the origin are placed so that the resistance between these planes is the same as the resistance in *a*, what is the distance from the origin to one of the planes?

3.13. The shunt field of a separately excited motor is wound with 5,000 turns of copper wire 32 mils in diameter. The average length of turn is 18 in. 220 volts are impressed across this field and after the motor has been running 8 hr. it is found that the average temperature of the shunt field winding is 100° C. (*a*) What current flows through the winding at this temperature? (*b*) What is the A.W.G. gauge number of the wire used?

3.14. No. 12 steel wire (American Steel Wire Gage) has a diameter of 0.1055 in. Its specific gravity is 7.78 and its resistivity is 13.8 microhm-cm. at 20° C. (*a*) What will be the length of this wire, in feet required to make a coil which will radiate 5 kw. at 80° C. when connected across 110-volt mains? (*b*) How many pounds of wire will be required?

3.15. The field coil of the motor described in Problem 3.13 has a resistance of 76.13 ohms after the motor has been standing idle at a room temperature of 20° C. After the motor has been running for several hours, the resistance of the field coil is found to be 106 ohms. What is the average temperature of this coil at the time the second measurement is made?

3.16. (*a*) A No. 10 A.W.G. copper wire has a resistance of approximately 1 ohm per 1,000 ft. What is the approximate resistance per 1,000 ft. of a No. 14 copper wire? (*b*) The No. 10 wire has a diameter of about 0.1 in. What is the approximate diameter of a No. 14 wire?

3.17. An electric furnace designed to operate at 220 volts gives a temperature of 500° C. above room temperature. Increasing the voltage 10 per cent is found to increase the temperature rise by 10 per cent. What change in the resistance of this furnace would produce the same increase in temperature rise without any change being made in the voltage?

3.18. (*a*) What is the percentage increase in resistance of copper per degree increase in temperature, referred to the resistance at 0°? (*b*) How many degrees increase in temperature is required to increase the resistance of copper 1 per cent?

3.19. A copper wire 1 ft. long and 0.001 in. in diameter has a resistance at 20° C. of 10.4 ohms. What will be the resistance at this same temperature of a coil of copper wire having 200 turns, if the diameter of the wire is 0.1 in. and the mean length of each turn in the coil is 22 in. (mean diameter 7 in.)?

3.20. A right circular cone of height h is split into two equal parts by a plane passed through the apex, perpendicular to the base. One-half of the cone is again split into two parts by a plane passed through it, parallel to the base and midway between the base and the apex. What is the resistance

of each of these parts along its length? The radius of the base of the cone is r.

3.21. A solid of revolution is formed by revolving the curve $x = y^2 + 1$ about the x-axis. If x and y are in centimeters and the solid of revolution has a resistivity of 100 ohms per mil ft.: (a) What is the resistance of this solid between two equipotential yz planes; one at $x_1 = 2$, and the other at $x_2 = 100$? (b) What will be the resistance of the solid between yz planes when $x_1 = 2$ and $x_2 = 1,000$? (c) Compare a and b.

3.22. Two concentric spherical shells of copper have diameters of 1 and 4 cm. They are separated by an insulating material having a resistivity of 10^6 ohm-cm. (a) What is the resistance of the path between the two spheres? (b) If the inner sphere is displaced slightly from its concentric position with respect to the outer sphere, is the resistance increased or decreased?

3.23. A solid of revolution is formed by rotating the curve

$$y = 2x^2 + 3x + 1$$

about the x axis. If x and y are measured in centimeters and the solid is made of material of resistivity of 100 ohms per mil ft., find the resistance between two equipotential planes perpendicular to the axis of revolution, one at the origin, the other at $x = 2$.

3.24. A semicircular copper annulus has a uniform rectangular cross section of 5 sq. cm., an internal radius of 10 cm., and an external radius of 15 cm. What is the resistance between the rectangular ends of the annulus if the resistivity of copper is 1.70 microhm-cm.? (Assume the equipotential surfaces are radial planes.)

3.25. A piece of sheet zinc $\frac{1}{4}$ in. thick is cut in the shape shown in the figure below, so that

$$y = b + ax^2$$

where $a = 0.01$, $b = 0.5$, and $l = 24$ in. The resistivity of the zinc is 5.8 microhm per centimeter cube at 20° C. Assume that the temperature coefficient is that of pure zinc. Compute as precisely as practicable the resistance of the zinc piece at a temperature of 50° C.
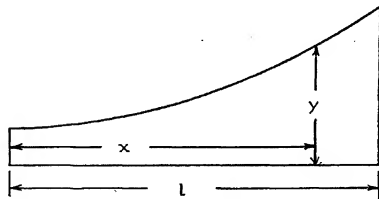
PROB. 3.25.

3.26. A two-wire transmission line connects a 600-volt generator with two motors. The first is located 1,000 ft. from the generator and takes 15 amp. The second is located 1,500 ft. from the generator; its current is 12 amp.

The allowable voltage drop in the line, from end to end, is 10 per cent of the generator voltage.   Determine, by an exact method, the most economical sizes of bare copper wire (minimum weight) to use for the two sections of the transmission line.   What size wires would you actually specify?

3.27. A certain transmission line uses copper-clad wire of 60 per cent conductivity at 20° C.   The wire is composed of steel having a 20 per cent conductivity and copper having 90 per cent conductivity at 20° C.   When 200 amp. flow through the wire, its total temperature rise is 50° C.

The specific heat of copper is 0.092 and of steel is 0.118.   The resistance temperature coefficient, per degree Centigrade at 20° C., of the copper is 0.0040 and of the steel is 0.0030.   The copper surface of the wire dissipates heat to the surrounding air at the rate of 1 watt per sq. ft. of surface per degree Centigrade temperature rise.

What is the diameter of the wire and of the steel core in mils?   Find the resistance of the wire at working temperature.   (Neglect temperature differences inside the wire.)

3.28. A submarine telephone cable is to be laid between two islands 5 miles apart.   The maximum depth of water occurs midway between the islands and is 2 miles.   The temperature of the water at its lowest point is 41° F. and may be assumed to vary as the square of the vertical distance above this point.   It reaches 68° F. at the surface.   For a first approximation, the shores may be considered to slope uniformly to the deep point. One of the circuits (two wires) in the cable is composed of No. 19 A.W.G. copper wire of 99 per cent conductivity, whose resistance temperature coefficient is 0.0040 at 20° C.   The loop resistance of this circuit was measured at a uniform temperature of 75° F.   How much will this loop resistance decrease when the cable is in place on the ocean floor?

3.29. A Wheatstone's bridge is modified to permit its use as a temperature measuring device.   The circuit diagram is shown below.   *A*, *B*, *C* are 1,000-ohm resistances that are constant regardless of temperature.   *D* is a coil of copper wire having a resistance of 1,000 ohms at 0° C. The temperature coefficient of resistance of the wire of *D* is 0.00427 per degree centigrade.   This coil is connected to the apparatus by flexible leads so that the coil may be placed in the space of which the temperature is to be measured.   *F*, *F* are 1,750-ohm resistances; *H* is a 6,500-ohm potentiometer.
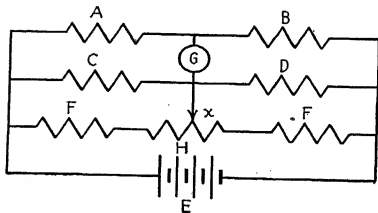
PROB. 3.29.

The coil is placed in the space of temperature $T°$ C. H is then adjusted so that the galvanometer reads zero.   Obtain a calibration curve for the apparatus, showing T as a function of x from $T = -50°$ C., to $T = 100°$ C.

3.30. The insulation resistance between the conducting core and the lead sheath of a cable is 5 megohms.   A non-varying potential difference of 11,000 volts is maintained between the core and the sheath.   (*a*) What is the conductance of this insulation?   (*b*) How much power is dissipated as heat in this insulation?

3.31. Current flows parallel to the axis of a piece of No. 0000 A.W.G. copper wire (a wire 0.46 in. in diameter).   The current density varies over the circular cross section of the wire in the following manner:

$$J = 10^6 + 4.36 \times 10^9 r^2 \qquad \text{amperes per square meter}$$

where r is the distance from the axis of the wire to the point at which the current density is J in. cm.

   a. What is the total current in the wire?

   b. If the resistivity of the copper is 1.75 microhm-cm., what is the rate of dissipation of heat in a piece of wire 1,000 ft. long?

   c. What is the resistance of such a length of wire under the conditions described above?

3.32. If the radius of the outer conductor of a concentric cable is R, calculate the radius of the inner conductor such that the voltage drop per m. (caused by leakage current) near the surface of the inner conductor is a minimum for an applied voltage V volts.

CHAPTER IV

INTRODUCTION TO TRANSIENT PHENOMENA IN ELECTRIC CIRCUITS

4.01. Introduction.—The magnitude of an electric current never changes instantaneously. For example, when the switch is closed in a single-mesh circuit comprising one or more circuit elements, one or more constant sources of emf, and the switch, the current increases continuously until its value approaches the magnitude calculated by the methods described in Chap. II. For this reason it is said that there are *transient* phenomena that may precede the *steady-state* conditions. Theoretically, the duration of the transient is infinitely long; from this point of view the term *steady-state* is meaningless. Practically, the duration of transient effects *having magnitudes greater than 1 per cent of the magnitudes of the corresponding steady-state effects* is usually less than a few milliseconds; from this point of view the term *steady-state* is commonly used to describe currents and voltages that become constant, or become periodic with constant amplitude, within a short interval of time after the operation that instigated them. Nevertheless, the student should remember that relatively long transients can be produced.*

The differential equations derived in this chapter and their solutions are the bases of electric-circuit theory. The mathematical manipulations required to obtain the solutions are at best tedious, and they are confusing to students who have not had long practice in their use. The emphasis therefore is placed on the presentation of the solutions and a discussion of their meaning from the physical point of view and not upon their derivation. In effect, the solutions are grabbed out of thin air. It is recommended that the student accept the solutions as (1)

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* Perhaps the most striking example of such phenomena is *superconductivity*. Current in a ring of lead (or other superconductive element, compound, or alloy at a temperature of a few degrees Kelvin), initiated by a changing magnetic field through the ring, has been observed to flow with an attenuation of the order of 1 per cent per hour.

mathematical relations that experiments show to be valid and (2) mathematical relations that satisfy the corresponding differential equations. He should choose numerical values for the constants of the solutions and plot representative graphs of the solutions in order to learn to correlate the mathematical relations and their graphical representations. Subsequently he can acquire by further study and practice the facility for solving the equations. He should not lose sight of two facts: (1) the differential equation is fundamental in circuit theory because its solution represents both transient and steady-state phenomena; and (2) the process of solving the differential equation is often so tedious, and the engineer is so often interested in the steady-state solution, that elaborate attempts are made to obtain the steady-state solution without obtaining the complete solution. Such methods are discussed for direct currents in Chap. II and for alternating currents in Chaps. VI and VII.*

When a current i in a loop of wire of resistance R increases continuously from zero at time $t = 0$ to some constant value I at time $t = T$, the energy input to the wire exceeds the integral $\int_0^T Ri^2 \, dt$. Thus during the transient interval, energy is transformed into heat *and* into some other form. The student will recall that there are magnetic effects associated with an electric current and that a transformation of energy is associated with these effects. The magnetic effects are observed outside of the circuit elements and they are usually correlated in terms of the magnetic field (see Chaps. XIII and XIV). *In this chapter the discussion is confined to the effects measured in the circuit.* Some of these effects are discussed in Sec. 1.13; they are discussed in detail in this chapter and in later chapters.

When a battery is connected to a pair of parallel plates separated by an insulator, a transient current flows through the battery. Some of the energy represented by this transient-current flow is assumed to be transformed into potential energy stored within the device comprising the two plates and the insulator between them. The student will recall that it is useful to describe the phenomena in this device (a condenser) in terms of the electric field (see Chap. XI). *In this chapter the discussion*

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* A brief analysis of methods for solving the differential equations derived in this chapter is given in Appendix A.

is confined to the effects measured in the circuit. Some of these effects are discussed in Sec. 1.13; they are discussed in detail in this chapter and in later chapters.

In order to discuss the transient phenomena considered in subsequent sections, the effects described in the last two paragraphs are attributed to *inductance* and *capacitance.* These two terms are discussed briefly in the two sections that follow. The concepts they represent are analyzed further in subsequent chapters.

4.02. Introductory Note on Inductance.—Whenever the current in a conductor changes with time there is an emf *induced* in the conductor. This emf is in many cases directly proportional at each instant to the *rate of change* of current at this instant, but not to the current itself, *viz.,*

$$(4.01) \qquad e = -L \frac{di}{dt} \qquad\qquad \text{volts}$$

Note that this experimental result shows that (1) if the current is increasing, the emf e is in the direction opposite to the direction of current flow and (2) if the current is decreasing (if di/dt is negative), the emf e is in the direction of the current. Thus the induced emf opposes any change in the current. The quantity L is called the *self-inductance* of the conductor. Equation (4.01) shows that it can be expressed in units "volt-seconds per ampere"; this unit is called the *henry.* The inductance of a straight conductor of given length is less than the inductance of the same conductor arranged in a closely wound coil. Coils of wire of various forms are called *inductors* or *choke coils.* Their inductances in a few cases are easily calculable (Chap. XIII).

Whenever the current in a conductor or coil changes with time there is an emf induced in each near-by conductor or coil. In many cases the magnitude of this emf is directly proportional to the rate of change of that current. For example, if i_1 is the current in one coil and i_2 the current in a second coil near the first,

$$(4.02) \qquad e_{12} = \pm M_{12} \frac{di_2}{dt} \qquad\qquad \text{volts}$$

$$(4.02a) \qquad\qquad _{21} \frac{di_1}{dt} \qquad\qquad \text{volts}$$

where the symbol e_{12} represents the emf induced in coil 1 by a

change in the current in coil 2, M_{12} is the *mutual inductance* of coil 1 with respect to coil 2, and e_{21}, M_{21} are defined analogously. Experiment shows that

$$(4.03) \qquad\qquad M_{12} = M_{21}$$

and theoretical considerations to be discussed later (Chap. XIV) correlate this fact and other relevant experimental data concerning the mutual mechanical forces that such current-carrying coils exert on each other.   Since the direction of an induced emf in one circuit is reversed if the direction of winding of *one* winding is reversed, the sign $(\pm)$ of the induced emf is not specified in Eq. (4.02).   Mutual inductance is conveniently expressed in henrys.

When one or more coils are wound on an iron core, the self- and mutual-inductances are in general far greater (as much as several thousand times larger) than the corresponding quantities associated with the same coils in the absence of the iron core. Furthermore the self- and mutual-inductances vary with the magnitude of the currents in coils wound on iron cores.   These effects are discussed subsequently (Chaps. XII and XIV).   *In this chapter it is assumed that L and M are constants.*

If a number of inductors $L_1, L_2, L_3, \ldots, L_n$ are connected in series so that the same current flows in all of them, the sum of the induced emfs, if the coils are so placed that their mutual inductances are negligibly small, is

$$(4.04) \quad e_1 + e_2 + e_3 + \cdots + e_n = -L_1 \frac{di}{dt} - L_2 \frac{di}{dt} - L_3 \frac{di}{dt}$$
$$\cdots - L_n \frac{di}{dt} \quad \text{volts*}$$

Thus the total induced emf e across the terminals of the series of inductors is

$$(4.05) \qquad\qquad e = \frac{di}{dt} \qquad\qquad \text{volts}$$

where

$$(4.06) \qquad L = L_1 + L_2 + L_3 + \cdots + L_n \qquad \text{henrys}$$

* Note that, if the inductors have mutual inductance with respect to each other, there will be a series of terms like $\pm M_{hk} \frac{di}{dt}$, in addition to those given in Eq. (4.04).

4.03. Introductory Note on Capacitance.—Whenever a difference of potential exists between two conductors separated by an insulator and distant from other conductors there is a charge on each conductor proportional to the difference of potential between the conductors. The charges are equal in magnitude but opposite in sign. From an experimental point of view the important facts are that for a given difference of potential:

1. The factor of proportionality between potential difference and charge depends upon the insulating material between the plates. It is, for example, about five times as great for glass as for air.

2. The factor of proportionality between potential difference and charge depends upon the geometric arrangement of the conductors. For example, if the distance of separation of the conductors is decreased, the proportionality factor increases.

These facts suggest that within the dielectric that separates the two conductors there are effects the nature of which cannot be simply deduced from the theoretical hypotheses thus far presented. It is convenient to postpone a detailed consideration of this problem (Chaps. X and XI).

The proportionality factor obtained by dividing the magnitude of the charge on either conductor by the magnitude of the difference of potential between the two conductors is called the *capacitance* C of the two conductors.

$$(4.07) \qquad\qquad = Cv \qquad\qquad \text{coulombs}$$

The practical unit of capacitance, the coulomb per volt, is called the *farad*. In most problems capacitances are of the order of a few millionths of a farad or less. They are therefore described in terms of a unit equal to 10^{-6} farad, which unit is called the *microfarad*.

Methods for calculating capacitance are given in Chap. XI. In all problems treated before Chap. XI it is assumed that the capacitance may be determined experimentally from the relation shown in Eq. (4.07).

Note that the current flowing up to one terminal of a capacitance is

$$(4.08) \qquad\qquad \frac{dq}{dt} \qquad \overline{dt} \qquad\qquad \text{amperes}$$

and an equal current flows away from the other terminal. *In most practical cases there is an appreciable conduction current through the dielectric so that Eq. (4.08) is, in a sense, incomplete. In this chapter it is assumed (1) that the capacitance is constant and (2) that there is no leakage current through the dielectric.*

If a number of capacitors C_1, C_2, C_3, $\cdots$, C_n is connected in parallel the same difference of potential v exists across all of the capacitors. The total charge q on the condensers is therefore

$$(4.09) \qquad q = C_1 v + C_2 v + C_3 v + \cdots + C_n v \qquad \text{coulombs}$$

Therefore the total capacitance C is

$$(4.10) \qquad C = \frac{q}{v} = C_1 + C_2 + C_3 + \qquad + C_n \qquad \text{farads}$$

If a number of capacitors are connected in series the difference of potential v across the series is

$$(4.11) \qquad \frac{q_1}{C_1} \qquad \frac{q_3}{} \qquad + \frac{q_n}{C} \qquad \text{volts}$$

The experiment that was described as a basis for the definition of capacitance shows that, if the charge on one electrode of C_1 is $+q_1$, the charge on the other electrode of C_1 is $-q_1$. But the electrodes and connecting wire between C_1 and C_2 were originally isolated and electrically neutral. Therefore after charging, the quantity of electricity on the electrode of C_2 that is connected to C_1 must be $+q_1$. Therefore

$$(4.12) \qquad q_1 = q_2 = q_3 = \cdots = q_n \qquad \text{coulombs}$$

The capacitance C of the series of capacitors through which there is no leakage conduction current may therefore be calculated

$$(4.13) \qquad \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdot \qquad \frac{1}{C} \qquad \text{per farad}$$

It is important to keep in mind the limitations, described in Secs. 4.01, 4.02, and 4.03, of the *introductory* theory of transients that is the subject of the remaining sections of this chapter. Only single-mesh circuits are discussed. The instantaneous voltage *drops* in these circuits are

1. The resistance drop v_R

$$(4.14) \qquad v_R = Ri \qquad \text{volts}$$

2. The drop attributed to self-inductance v_L

$$(4.15) \qquad\qquad v_L = L \frac{di}{dt} \qquad\qquad \text{volts}$$

3. The drop in potential v_C between the terminals of a condenser in which there is assumed to be no leakage

$$(4.16) \qquad\qquad\qquad\qquad\qquad \text{volts}$$

The parameters R, L, C are assumed to be constants. The total drop of potential between the terminals of an inductor is the sum of a term like Eq. (4.14) and a term like Eq. (4.15). In the sections that follow, all the series-resistance terms, representing the resistances of inductors, resistors, wires, and sources of emf, are represented by the sum

$$(4.17) \qquad\qquad R = R_1 + R_2 + \ \ldots \qquad\qquad \text{ohms}$$

4.04. Transient and Steady-state Currents in RC Circuit.— Experiment shows that the current in a circuit comprising a
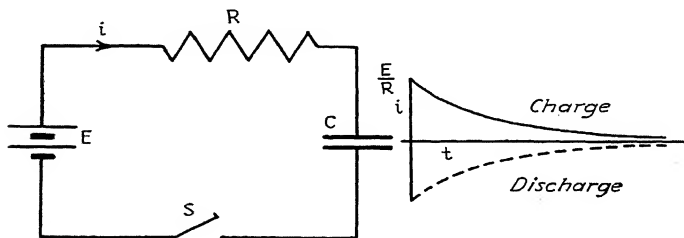
Fig. 4.01.

condenser, resistor, and battery connected in series varies (approximately) as shown in the oscillogram, Fig. 4.01. The current at the instant of closure of the switch ($t = 0$) is zero; it rises quickly (but not instantaneously) to the value E/R because the inductance of the conductors in the circuit is very small. The current then decreases exponentially, approaching zero asymptotically.

If, after the current becomes practically zero, the terminals of the condenser are disconnected from the circuit of Fig. 4.01 and connected directly to the resistance R, a current like that shown by the dotted curve in Fig. 4.01 flows in the resistor,

which current is in the opposite sense to that of the charging current.

These results may be expressed in terms of electric-circuit theory as follows. At any instant after the switch is closed (Fig. 4.01) there are three voltages in the circuit: E the rise of voltage through the battery, Ri the voltage drop through the resistor, and q/C the voltage drop through the condenser. According to Kirchhoff's second law, the algebraic sum of the voltage drops around the circuit is zero. Thus

$$(4.18) \qquad E = Ri + \frac{q}{C}$$

or

$$(4.18a) \qquad E = R\frac{dq}{dt} + \frac{q}{C}$$

If the charge on the condenser is zero at the instant of closure of the switch,* the solution of this equation is

$$(4.19) \qquad q = CE(1 - \varepsilon^{-\frac{t}{RC}}) \quad \text{coulombs} \quad (q = 0;\ t = 0)$$

and the current is

$$(4.20)\dagger \qquad i = \frac{E}{R}\varepsilon^{-\frac{t}{RC}} \quad \text{amperes} \quad (q = 0;\ t = 0)$$

When the charged condenser is discharged through the resistance

$$R\frac{dq}{dt} + \frac{q}{C} \qquad \text{and} \qquad q = CE \qquad \text{at } t = 0$$

so that

$$(4.21) \qquad q = CE\varepsilon^{-\frac{t}{RC}} \qquad \text{coulombs}$$

and

$$(4.22) \qquad i = \frac{E}{R}\varepsilon^{-\frac{t}{RC}} \qquad \text{amperes}$$

* See Problem 4.19 on page 133 for result when $q = Q_0$ at $t = 0$.
† Note that the solution gives $i_0 = E/R$ at $t = 0$, contrary to the statement in the first paragraph of this section. This result is obtained because it has been assumed for the theoretical solution that the inductance in the circuit is zero, which is never true.

The minus sign indicates that the current during discharge flows in the sense opposite to the assumed sense of the charging current.

From Eq. (4.20) the energy output of the battery during the *charging* period is

$$(4.23) \qquad W_1 = \int_0^\infty Ei\, dt = \frac{E^2}{R} \int_0^\infty \varepsilon^{-\frac{t}{RC}}\, dt = CE^2 \qquad \text{joules}$$

while the energy output from the condenser during the discharging period is from Eq. (4.22)

$$(4.24) \qquad = \int_0^\infty Ri^2\, dt = \frac{E^2}{R} \int_0^\infty {}^{2t} \qquad \text{joules}$$

Thus the energy stored in the condenser is $\frac{1}{2}CE^2$ joules, and during the charging periods equal amounts of energy ($\frac{1}{2}CE^2$) are transformed into heat by the resistance R and into energy stored in the condenser. Note that the time $t = RC$ is commonly called the *time constant* of the circuit.

The methods for calculating the steady-state currents in networks to which sources of alternating emf are connected are discussed in Chaps. V, VI, VII. Both the transient and steady-state currents in a single-mesh circuit can be calculated in a few simple cases from the differential equations that represent the circuit. Examples of this procedure are presented in this chapter. In these examples it is assumed that the alternating emfs are sinusoidal functions of time, *i.e.*, that such an emf is represented mathematically

$$(4.25) \qquad\qquad E \cos (\omega t + \qquad\qquad \text{volts}$$
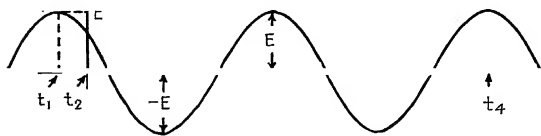
Fig. 4.02.

or by a sine function of similar form. This implies that E is the peak voltage because the maximum and minimum values of $\cos (\omega t + \phi)$ are ± 1; that $(\omega t + \phi)$ is a number expressed in radians that varies linearly with t (it is assumed that ω and ϕ are constants). When $(\omega t + \phi)$ varies linearly with t from

$(\omega t + \phi) = 0$ to $(\omega t + \phi) = 2\pi$, the graph of e is one complete cycle of a cosine wave, as shown in Fig. 4.02. Successive equivalent cycles are traversed as $(\omega t + \phi)$ varies from 2π to 4π, from 4π to 6π, etc.

According to the notation shown in the figure

$$(4.26) \qquad\qquad t_2 - t_1 = \frac{\phi}{} \qquad\qquad\qquad \text{seconds}$$

$$(4.27) \qquad\qquad - t_1 = t_4 - t_3 = \frac{2\pi}{} \qquad\qquad \text{seconds}$$

Thus ϕ is the interval in seconds between the instant for which the voltage is maximum and the instant for which the time is zero; ϕ is often called the *phase angle* of the alternating emf with respect to $t = 0$. The interval in seconds required for the voltage to traverse a complete cycle $(t_3 - t_1)$ or $(t_4 - t_3)$ is called the *period* of the voltage. Since the period is equal to the reciprocal of the number of cycles per second or *frequency f* of the voltage, it follows that

$$(4.28) \qquad\qquad \omega = 2\pi f \qquad\qquad \text{radians per second}$$

These concepts are discussed in detail in Chap. V.

A source of alternating voltage e, a capacitance C, and a resistance R are shown connected in series in Fig. 4.03. The problem is: If the switch is closed at the instant $t = 0$ find the sub-
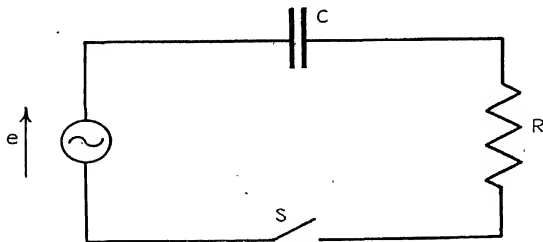
Fig. 4.03.

sequent variation of the current with time. The differential equation relating the charge q and the time t is like Eq. (4.18a) except that the constant emf E of that equation is replaced by the sinusoidal emf $E \cos (\omega t + \phi_1)$:

$$(4.29) \qquad\qquad E \cos (\omega t + \phi_1) = R \frac{dq}{dt} + \frac{q}{C}$$

Thus the solution in this case corresponds to Eq. (4.21)

$$(4.38) \qquad = \frac{E}{R} \, \varepsilon \qquad\qquad \text{amperes}$$

From Eq. (4.38) the energy W associated with an inductance L carrying a current $I = E/R$ is

$$(4.39) \qquad = \int_0^{\epsilon} \qquad = \frac{E^2}{R} \, \Big|_0^{\infty} \, \varepsilon^{-\frac{2R}{L}t} \, dt = \frac{1}{2} L I^2 \qquad \text{joules}$$

This energy, which is assumed to represent the energy required to produce the magnetic field surrounding the current-carrying
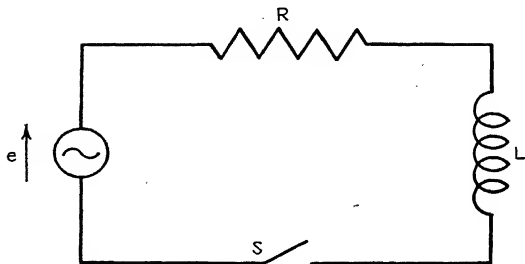
FIG. 4.05.

inductor, is analyzed further in subsequent chapters (particularly Chap. XIII). Note that in this case L/R is called the *time constant* of the circuit.

If a source of sinusoidal alternating current is connected in series with an inductance L and a resistance R as shown in Fig. 4.05, the differential equation relating the current and time is

$$(4.40) \qquad E \cos (\omega t$$

where ϕ_1 is the phase of the voltage at the instant the switch is closed. This equation is of the same form as Eq. (4.29), so that the solution, for $i = 0$ at $t = 0$, may be written down immediately by substituting i for q, L for R, and R for $1/C$ in Eqs. (4.30), (4.31), (4.32), and (4.33)

(4.41) $+ B_1 \cos (\omega t + \phi)$ amperes

(4.42) E $\cos \phi$ amperes

(4.43) E amperes

and

(4.44) radians

Thus there is a transient term that decreases with time $(\varepsilon^{-\frac{R}{L}t})$ superimposed upon a sinusoidal current. When the former has
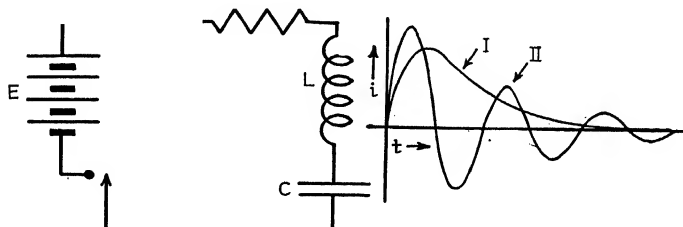
FIG. 4.06.

become negligibly small, the latter becomes the steady-state current.

Note that if the switch is closed at an instant such that

(4.45) radians

the coefficient of the transient term is zero, *i.e.*, the steady-state sinusoidal current, without the transient, flows after the switch is closed.

4.06. Transient and Steady-state Currents in RLC Circuit.— Experiments show that the current in a circuit consisting of a resistance, inductance, capacitance, and battery, shown in Fig. 4.06, may vary cyclically with time (Curve II) or it may be a single pulse (Curve I). The wave form of the current depends upon the magnitudes of R, L, and C, as shown in the discussion below.

If the current varies cyclically with time, the circuit is often called an *oscillating circuit*.   The time for one complete oscillation or *period* is constant although the peak value or *amplitude* of the current decreases with time.   By connecting an oscillating circuit to a vacuum tube (Chap. XV), *sustained oscillations* of constant amplitude (*i.e.*, alternating currents) can be produced.   Such vacuum-tube oscillators are used to produce the high-frequency alternating currents required for the transmission (and in some cases, for the reception) of radio and television signals.   The oscillations of the kind shown in Fig. 4.06 (Curve II) are called *damped* oscillations, the implication being that they decrease with time, finally dying out.

The differential equation representing the time variation of the charge on the condenser when the switch is thrown to the left (Fig. 4.06) at $t = 0$ is

$$(4.46) \qquad\qquad E = \qquad\qquad R\frac{dq}{dt}$$

This second-order linear equation with constant coefficients can be solved by the methods discussed in Appendix A.   In this case the *particular solution* is a constant CE, which, by inspection, satisfies Eq. (4.46).   The implication is that, whatever may occur during the transient interval, the circuit finally arrives at a state in which there is an emf E in the battery and a charge $Q = CE$ on the condenser, no current flows, the rate of change of current with time is zero, and the net voltage drop going clockwise around the circuit is $-E$ through the battery plus zero through R plus zero through L plus E through C, or zero.   Theoretically this state is attained an infinite time after the switch is closed, though practically the charge on C is immeasurably smaller than $Q = CE$ within a few seconds at most, and often within a few microseconds, after the switch is closed.

The form of the complementary solution (the transient) depends upon the relative magnitudes of R, L, and C.   The first step toward obtaining the transient terms is to write the *auxiliary equation* by substituting p for d/dt and p^2 for d^2/dt^2 and then solving for p, assuming* that it is an algebraic quantity

* The validity of this assumption can be checked by substituting the solutions obtained from it into the differential equation (4.46).

$$(4.47) \qquad \left(Lp^2 + Rp + \frac{1}{C} \right) q = 0$$

so that, if q is not always zero,

$$(4.48) \qquad p = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \equiv \alpha \pm \mu$$

There are three conditions, leading to solutions of different forms:

1. $\dfrac{R}{2L} = \dfrac{1}{\sqrt{LC}}$; $\mu = 0$.

2. $\dfrac{R}{2L} > \dfrac{1}{\sqrt{LC}}$; μ is real.

3. $\dfrac{R}{2L} < \dfrac{1}{\sqrt{LC}}$; μ is imaginary.

The complementary solution for $\mu = 0$ is

$$(4.49) \qquad q = C_1 \varepsilon^{\alpha t} + C_2 t \varepsilon^{\alpha t} \qquad \text{coulombs}$$

The complete solution obtained by adding the complementary solution and the particular solution is, for the conditions $q = 0$ and $i = 0$ at $t = 0$

$$(4.50) \qquad q = CE[\varepsilon^{\alpha t}(\alpha t - 1) + 1] \qquad \text{coulombs}$$
$$(4.51) \qquad i = CE\alpha^2 t \varepsilon^{\alpha t} \qquad \text{amperes}$$

For the case when μ is real, the charge and current are

$$(4.52) \qquad q = \frac{CE}{\mu} \varepsilon^{\alpha t}(\alpha \sinh \mu t - \mu \cosh \mu t) + CE \qquad \text{coulombs}$$

$$(4.53) \qquad i = \frac{E}{\mu L} \varepsilon^{\alpha t} \sinh \mu t \qquad \text{amperes}$$

When μ is imaginary, the solutions can be obtained from Eqs. (4.47) and (4.48) by writing $j\omega_0$ for μ, where $j = \sqrt{-1}$, and reducing the results by introducing the equivalence of $-j \sinh j\omega_0 t$ and $\sin \omega_0 t$. There results

$$(4.54) \qquad q = CE \left[1 + \frac{\sqrt{\alpha^2 + \omega_0^2}}{\omega_0} \varepsilon^{\alpha t} \sin (\omega_0 t + \phi) \right] \qquad \text{coulombs}$$

where

$$\phi = -\tan^{-1} \frac{\omega_0}{\alpha}$$

and

$$(4.55) \qquad i = \frac{E}{\omega_0 L} \, \varepsilon^{\alpha t} \sin \omega_0 t \qquad \text{amperes}$$

This then is the case that gives rise to an oscillating current. The *natural frequency of oscillation* is

$$(4.56) \qquad f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \quad \text{cycles per second}$$

If the condenser is charged to a voltage E and then connected to R and L in series, the charge on the condenser varies as shown by Eq. (4.54) without the constant term (CE); the current corresponds to Eq. (4.55). The phenomenon of oscillation in this circuit can be usefully described qualitatively. When the switch is closed, the energy stored in the condenser $(\frac{1}{2}CE^2)$ is the sole source of energy in the circuit. Subsequently the current increases, the energy stored in the condenser decreases, the energy represented by current flow in the inductance increases, and some energy is dissipated by the resistance in the form of heat. This is followed by a period during which the energy represented by current flow through the inductance $(\frac{1}{2}Li^2)$ decreases, the energy stored in the condenser increases, and more energy is lost as heat. Thus there is a cyclic exchange of energy between the condenser and the inductance, with the resistance continuously decreasing the available energy by transforming it into heat (Ri^2). This accounts for the damping of the current wave. If there were no resistance (a condition impossible to obtain), the charge and current would be

$$(4.57) \qquad q = CE \sin\left(\frac{t}{\sqrt{LC}} - \frac{\pi}{2}\right) \qquad \text{coulombs}$$

$$(4.58) \qquad i = \sqrt{\frac{C}{L}} \, E \sin \frac{t}{\sqrt{LC}} \qquad \text{amperes}$$

and the current would continue to flow forever with a frequency

$$(4.59) \qquad f_0 = \frac{1}{2\pi \sqrt{LC}} \qquad \text{cycles per second}$$

When an RLC circuit is connected to a source of sinusoidal emf, as shown in Fig. 4.07, the current consists of a transient

term like one of those described above superimposed upon a steady-state sinusoidal current. The differential equation for the circuit is

$$(4.60) \qquad \dot{E} \cos (\omega t + \phi_1) = L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C}$$

The general solution for the current is

$$(4.61) \quad i = (\alpha + \mu)C_1 \varepsilon^{\mu t} + (\alpha - \mu)C_2 \varepsilon^{-\mu t}$$
$$+ \frac{E}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}} \sin (\omega t - \theta + \phi_1) \quad \text{amperes}$$

where C_1 and C_2 are undetermined constants depending upon boundary conditions, and

$$(4.62) \qquad \theta = -\tan^{-1} \frac{\omega RC}{1 - \omega^2 LC} \qquad \text{radians}$$

The first two terms of Eq. (4.61) have been discussed above. Note that, if $\omega L = 1/\omega C$, the steady-state current is

$$(4.63) \qquad i = \frac{E}{R} \cos (\omega t + \phi_1) \qquad \text{amperes}$$
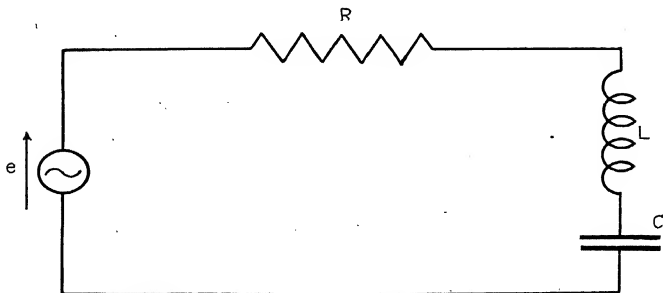
Fɪɢ. 4.07.

Thus the current is equal to the voltage divided by the constant R. This condition is one of several called *resonance*. For the particular case in which $\omega L = 1/\omega C$, the steady-state current is that which would flow if L and C were not in the circuit.

4.07. Limitations of the Material of This Chapter.—If the student wades through this chapter and comes up with a reasonably complete understanding of the phenomena that are described, he may acquire the opinion that he knows a great deal about circuit theory. It is the purpose of this section to puncture this bubble of false pride.

No physical theory correlates all facts, known and unknown, associated with a particular group of phenomena. If it did, all facts would be known and that branch of physics for which the theory was devised would be complete. Such a circumstance has never occurred. In particular the following considerations have been ignored in this chapter:

1. The discussion is confined to single-mesh circuits. A set of simultaneous differential equations is required for networks containing several meshes. The solutions are in general obtained by straightforward methods.[1]

2. The condenser in each circuit in which C is connected is assumed to have perfect insulation, *i.e.*, zero conductance or infinite resistance. Although this is never true in practice, the approximation obtained by assuming no leakage is often useful.

3. Very simple boundary conditions are assumed. Other boundary conditions are theoretically permissible and practically obtainable. They give rise to solutions not presented in this chapter, although some of these solutions are discussed in Appendix A.

4. Only constant and sinusoidal emfs are discussed. Many other wave forms are obtainable.

5. The coefficients $R, L, 1/C$, are here assumed to be constant. If they vary with time, the solutions of the differential equations can best be obtained by step-by-step integration[2] or by means of the differential analyzer,[3] which is a mechanical device for solving ordinary differential equations.

6. The effects of varying temperatures within the circuit and the effects produced by the interconnection of the different materials of which the circuit elements may be made are ignored. These effects are often negligible from the practical point of view.

7. Under certain conditions accompanying a rapid change of current in a circuit, electromagnetic energy is radiated from the circuit and does not return to it.[4] It is assumed here that

measurable effects in the circuit account for all of the energy, *i.e.*, that the circuit is a conservative system.

8. The problems discussed here can be solved by the methods of the *operational calculus*,[5] which is not considered here. The student will probably learn these methods in his courses in advanced circuit theory.

9. A complicated but useful nomenclature has been developed for designating many of the terms of the solutions presented in this chapter. Only a few of these names are presented in this chapter, while others are discussed in subsequent chapters.

10. The parameters R, L, C used in this chapter are called *lumped parameters*, *i.e.*, it is assumed that two-terminal devices, each containing only resistance, inductance, or capacitance, accurately represent the actual circuit elements. In many a-c circuits, and particularly in problems involving ultra-high-frequency alternating currents,* the parameters must be considered as *distributed parameters*. In such cases—for example, in a transmission line—it is necessary to consider a device in a circuit as one in which the resistance, inductance, and capacitance are (uniformly or nonuniformly) distributed from point to point in the device.

The equations of electric-circuit theory discussed in this chapter are linear differential equations with constant coefficients. These equations have the property that, if each of two or more functions satisfies a linear differential equation with constant coefficients, the sum of these functions also satisfies the equations. *Thus if several alternating emfs of different frequencies (including perhaps zero) are applied to a network simultaneously, the solutions may be obtained separately and then added to obtain the complete solution.* This inference from the mathematical theory is of great practical importance. It is called the *principle of superposition*, an extension of the theorem of Sec. 2.03.

4.08. Illustrative Problem.

Problem.—The terminal voltage of a generator varies with time as shown in Fig. 4.08.† This generator is connected in series with a 1,000-ohm resistor and a 0.1-μf condenser. Calculate the voltage across the resistor

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* For present purposes, ultra-high frequencies are frequencies greater than 300 megacycles per second.

† Such generators—usually vacuum-tube oscillators—are commonly used to test a-c circuits, particularly vacuum-tube amplifiers (see Chap. XV)

and the voltage across the condenser for a time interval of 0.001 sec., after the circuit has been operating long enough so that the "steady state" has been reached.

Solution.—When "steady state" is attained, the problem can be treated as two interdependent transient problems. During an interval AB the condenser is being charged in one sense by a constant voltage $V = 10$ volts; during an interval BC the condenser is being charged in the opposite sense by a constant voltage $-V = -10$ volts. The charge on the condenser just
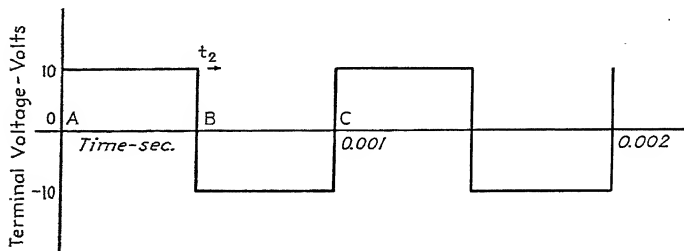
Fig. 4.08.

before $t_1 = 0$, *i.e.*, at the *end* of the charging period for $-V = -10$ volts, must be equal to the charge on the condenser at the instant the voltage becomes $V = 10$ volts. Similarly, the charge when $t_1 = T/2 = 5 \times 10^{-4}$ sec. must be equal to the charge for $t_2 = 0$. Using R as the symbol for the resistance, C for capacitance, and q_1 and q_2 for the charges on the condenser within the intervals $0 \leq t_1 \leq T/2$ and $0 \leq t_2 \leq T/2$, these conditions must satisfy the following differential equations:

(4.64)   0

(4.65) $0 :\quad \leq \dfrac{T}{2}$

(4.66)   coulombs

(4.67)   coulombs

The solutions of Eqs. (4.64) and (4.65) are

(4.68)
$$q_1 = CV(1 - \varepsilon^{-\frac{t_1}{RC}}) + Q_1\varepsilon^{-\frac{t_1}{RC}} \qquad 0 \leq t_1 \leq \frac{T}{2}$$

$$q_2 = CV(1 - \varepsilon^{-\frac{t_2}{RC}}) + Q_2\varepsilon^{-\frac{t_2}{RC}} \qquad 0 \leq t_2 \leq \frac{T}{2}$$

The boundary conditions of Eqs. (4.66) and (4.67) require that

$$(4.70) \qquad Q_1 = CV(1 - \varepsilon^{-\frac{T}{2RC}}) + Q_1\varepsilon^{-\frac{T}{2RC}}$$

$$(4.71) \qquad Q_1 = -CV(1 - \varepsilon^{-\frac{T}{2RC}}) + Q_2\varepsilon^{-\frac{T}{2RC}}$$

The sum of Eqs. (4.70) and (4.71) is

$$(4.72) \qquad (Q_1 + Q_2) = (Q_1 + Q_2)\varepsilon^{-\frac{T}{2RC}}$$

Since $\varepsilon^{-\frac{T}{2RC}}$ is *not*, in general, zero, it follows that

$$(4.73) \qquad Q_1 = -Q_2$$

as might have been predicted from the symmetry of the applied voltage. Solution of Eq. (4.70) or (4.71) with Eq. (4.73) shows that

$$(4.74) \qquad Q_1 = CV \frac{\varepsilon^{-\frac{T}{2RC}} - 1}{\varepsilon^{-\frac{T}{2RC}} + 1} \qquad \text{coulombs}$$

$$(4.75) \qquad Q_2 = CV \frac{1 - \varepsilon^{-\frac{T}{2RC}}}{1 + \varepsilon^{-\frac{T}{2RC}}} \qquad \text{coulombs}$$

By substitution of these values in Eqs. (4.68) and (4.69), the solutions are obtained:

$$(4.76) \qquad q_1 = CV \left[1 - \frac{2\varepsilon^{-\frac{t_1}{RC}}}{\varepsilon^{-\frac{T}{2RC}} + 1} \right] \qquad 0 \le t_1 \le \frac{T}{2}$$

$$(4.77) \qquad i_1 = \frac{dq_1}{dt_1} = \frac{2V}{R} \frac{\varepsilon^{-\frac{t_1}{RC}}}{\varepsilon^{-\frac{T}{2RC}} + 1} \qquad 0 \le t_1 \le \frac{T}{2}$$

$$(4.78) \qquad q_2 = -CV \left(1 - \frac{2\varepsilon^{-\frac{t_2}{RC}}}{\varepsilon^{-\frac{T}{2RC}} + 1} \right) \qquad 0 \le t_2 \le \frac{T}{2}$$

$$(4.79) \qquad i_2 = \frac{dq_2}{dt} = -\frac{2V}{R} \frac{\varepsilon^{-\frac{t_2}{RC}}}{\varepsilon^{-\frac{T}{2RC}} + 1} \qquad 0 \le t_2 \le \frac{T}{2}$$

Values of the condenser voltage q_1/C and resistance drop Ri_1 when $V = 10$ volts, $T = 10^{-3}$ sec., $R = 1,000$ ohms, $C = 10^{-7}$ farad are tabulated on page 130:

t_1, sec.	q_1/c, volts	Ri_1, volts
0	−9.867	19.867
2×10^{-5}	−6.265	16.265
4×10^{-5}	−3.317	13.317
6×10^{-5}	−0.903	10.903
8×10^{-5}	1.074	8.926
10^{-4}	2.691	7.309
2×10^{-4}	7.312	2.688
3×10^{-4}	9.011	0.989
4×10^{-4}	9.636	0.364
5×10^{-4}	9.867	0.133

The results (q_2/C and Ri_2) for the interval $0 \leq {}_2 \leq T/2$ are precisely the negatives of those tabulated for q_1/C and Ri_1. The results are shown in the curves of Fig. 4.09.
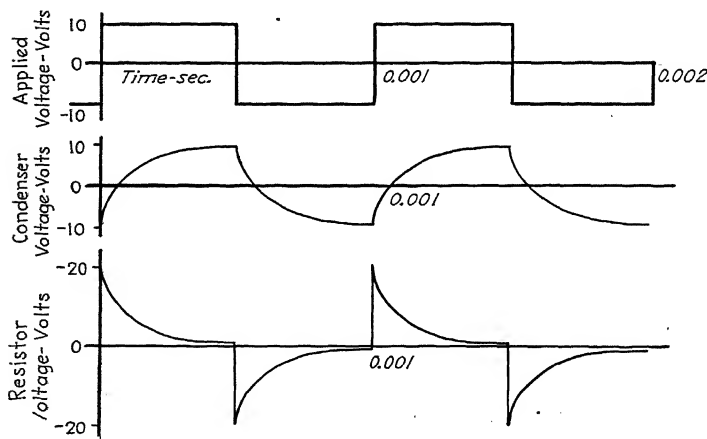
FIG. 4.09.

References

1. *Differential Equations;* H. T. H. Piaggio; pp. 42–48; George Bell & Sons, Ltd., London, 1931.
2. *Interpolation;* P. F. Steffensen, pp. 170–177, The Williams & Wilkins Company, Baltimore, 1927.
3. *The Differential Analyzer;* Vannever Bush; *Journal of the Franklin Institute;* 212, 447–488, 1931.
4. *High Frequency Alternating Currents;* Knox McIlwain and J. G. Brainerd; pp. 342–426; John Wiley & Sons, Inc., New York, 1931

5. *Electric Circuit Theory and the Operational Calculus;* John R. Carson; McGraw-Hill Book Company, Inc., New York, 1926.

For a concise but detailed discussion of transients in single- and two-mesh circuits with various boundary conditions and excellent reproductions of oscillograms superposed on graphs of analytical results, see *Introduction to Electrical Transients;* E. B. Kurtz and G. F. Corcoran; John Wiley & Sons, Inc., New York, 1935.

Problems

4.01. Calculate the equivalent capacitance of the circuit, the charge on each condenser, and the voltage between the terminals of each condenser.

4.02. A 100-μf condenser charged to 600 volts is discharged through a resistance of 5,000 ohms. Find (*a*) the current at the instant at which the current is decreasing at the rate of 0.2 amp. per sec., (*b*) the charge on the condenser when its stored energy is 2 joules, (*c*) the condenser voltage when it is discharging at the rate of 0.05 coulomb per sec., and (*d*) the rate at which the condenser is losing energy when its charge is 10,000 microcoulombs.

4.03. Two condensers having capacitances of 2μf and 1μf and initial terminal voltages of 300 and 500 volts are connected in series so that the terminal voltage of the combination is 800 volts. Calculate the voltage across each condenser if the combination is connected to a 100-volt battery.

4.04. The contact spring C is caused by a mechanical device to be held alternately in contact with A and with B for 0.05 sec. This cycle requires 0.1 sec. (10 cycles per sec.), *i.e.*, the times of travel of C from A to B and from B to A are assumed to be negligibly small. The meter MA indicates, under the conditions of this experiment, the average value in amperes of the current flowing through it. Calculate the steady meter reading.
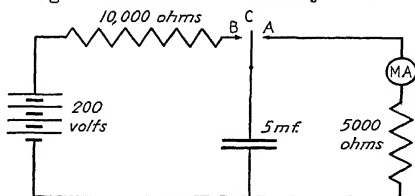
PROB. 4.01.

PROB. 4.04.

4.05. A ballistic milliampere-second meter having a resistance of 1,000 ohms is shunted by a 1,000-ohm resistor. This combination is connected in series with a 1,000-volt battery, a 20-μf condenser, a 9,500-ohm resistor, and a switch. If the condenser is initially uncharged, what maximum deflection of the meter is observed after the switch is closed?

4.06. Two circuits have resistances R_1 and R_2, self-inductances L_1 and L_2, and mutual inductance M. One of them is connected to a battery having
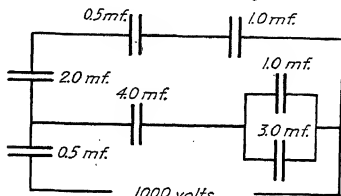
negligible internal resistance and emf E by closing a switch.   Show that the total quantity of electricity Q_2 that flows in the second circuit is

$$Q \qquad EM$$

4.07. An RL circuit is carrying a current I produced by an emf E.   At $t = 0$, E is disconnected and the RL circuit is short-circuited simultaneously. Calculate the time T at which the energy associated with the current-carrying inductance L is precisely equal to the energy dissipated by the resistor R from $t = 0$ to $t = T$.

4.08. The current in an RL circuit is increasing at the rate of 10 amp. per sec. when it is initially connected to a constant emf.   The final current is 1.6 amp.   What is the magnitude of the current 0.1 sec. after the switch is closed?

4.09. A 24-volt battery in series with a 12-ohm resistor is connected to a second 12-ohm resistor; across the latter a coil of resistance 4 ohms and inductance 0.03 henry is connected at $t = 0$.   Plot the current through the coil as a function of time.   Plot for comparison the current that would have been obtained if the first resistor in series with the battery had been left out of the circuit.

4.10. A 50-g. mass is suspended from one end of a spring of negligibly small mass; the other end of the spring is fastened to a fixed support.   If a second 50-g. mass is attached to the first, the extension of the spring increases 4 cm.   The single 50-g. mass, attached to the spring, is pulled downward 2 cm. and then released.   Calculate the velocity of a point in the 50-g. mass as a function of time, noting carefully all assumptions made in the course of the solution.

4.11. If a condenser of capacitance C, charged to a potential difference V volts, is discharged through a coil of resistance R and inductance L, calculate the time interval T between the initiation of the discharge and the instant at which the current in the circuit is maximum.

4.12. If a condenser of capacitance 0.2 μf, charged to a potential difference of 500 volts, is discharged through an inductance of 1 henry and a resistance of 3,000 ohms, calculate the frequency of the trans ent oscillation and the ratio of successive positive peaks of the current.   What is the resonant frequency of the circuit if the resistance is made negligibly small?

4.13. A condenser of capacitance C and leakage conductance G is connected in series with a resistance R, an emf E, and a switch.   What happens after the switch is closed?

4.14. The field winding of a motor has an inductance of 30 henrys and a resistance of 120 ohms.   If this winding is connected to a source of direct current by closing a switch, calculate the time interval required for the current to build up to 95 per cent of its final value.

4.15. An emf of 30 volts is connected at $t = 0$ to a resistance of 0.5 ohm and an inductance of 0.2 henry; at $t = 0.1$. sec the emf is replaced by a short-circuiting resistance of 0.1 ohm.   Calculate the magnitude of the current for $t = 0.18$ sec.

4.16. A relay coil, having a resistance of 15 ohms and an inductance of 2.5 henrys, is connected in series with a resistance of 10 ohms and a 50-volt

battery. If the current through the coil rises to 1.8 amp., the armature of the relay operates, shunting a 2-ohm resistor across the coil; if the current through the coil decreases to 0.7 amp., the armature is released. Will the armature vibrate? If it vibrates, what is the frequency of vibration?

4.17. If the current through the battery rises almost instantaneously to a value that remains constant thereafter, what is the current through each of the branches, and what relations exist among R_1, R_2, L, C?
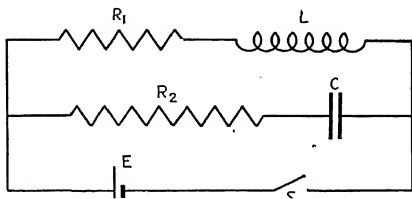
PROB. 4.17.

4.18. A voltage regulator alternately closes and opens a switch connected across a 40-ohm resistor that is in series with a 250-volt source and the field winding of a generator; the field has an inductance of 30 henrys and a resistance of 50 ohms. If the regulator operates at a frequency of 5 cycles per sec., and the switch is closed four times as long as it is open, calculate the average, the maximum, and the minimum values of the field current.

4.19. The neon tube N shown in the diagram below has the following characteristics: its capacitance is negligibly small compared to the capaci-
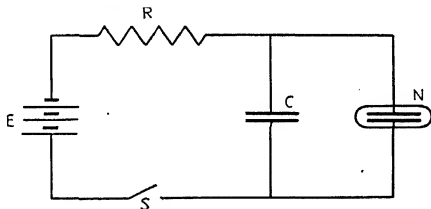
PROB. 4.19.

tance C; if the voltage across its terminals is increased from less than 52 volts, no current flows through it until the voltage rises above 65 volts; when current flows through it, its resistance can be assumed to be 50,000 ohms (constant) with negligible practical error; after current starts flowing in the tube, the current stops only if the terminal voltage becomes less than 52 volts. Describe the current which flows in the neon tube N if $E = 100$ v., $R = 500,000$ ohms, $C = 0.1\mu f$, and the switch is closed at $t = 0$.

4.20. If an RLC series circuit is connected by means of a switch to $e = E \sin (\omega t + \phi_1)$, find a set of boundary conditions and a value of ϕ_1 for which the transient current in the circuit is zero.

CHAPTER V

SINUSOIDAL ALTERNATING CURRENTS

5.01. Introduction.—The concept of a sinusoidal alternating emf or current and its mathematical representation were outlined briefly in Sec. 4.04. It is the purpose of this chapter to discuss in more detail the reasons for using alternating currents and the nomenclature that has been devised for designating some of their properties.

As noted in Chap. IV, an alternating current of frequency f cycles per second flows in a circuit connected to a source of an alternating emf of that frequency. When the current in the circuit is i and the potential drop between two points in the circuit at the same instant is v, the energy input to that part of the circuit in an infinitesimal time interval dt is $vi\,dt$ joules. The power input to that part of the circuit at that instant is $p = vi$ watts. Although the average value of an alternating voltage or current throughout an integral number of cycles is zero, the average value of the power input (or output) of an a-c circuit is *not* zero. Thus alternating current as well as direct current can be used to transmit electric energy. These points are discussed in detail in subsequent sections of this chapter.

5.02. Reasons for Using Alternating Currents.—The chief advantages which have led to the nearly universal use of alternating current for the generation, transmission, and use of electric power are:

1. A two-winding a-c transformer consists of a *primary coil* and a *secondary coil* wound on an iron core. Experiment shows that, when the primary is connected to a source of sinusoidal emf and the secondary is connected to a load, approximately sinusoidal currents flow in the primary circuit and in the secondary circuit in a well-designed transformer. Under certain conditions, the number of turns of wire in the primary coil N_1, the number of turns in the secondary N_2, the amplitudes of the primary and secondary voltages V_1, V_2, the amplitudes of the primary and

secondary currents I_1, I_2, the input power P_1, and the output power P_2 satisfy the following relations *approximately:*

$$(5.01) \qquad \frac{V_1}{V_2} \approx \frac{N_1}{N_2}; \qquad \frac{I_1}{I_2} \approx \frac{N_2}{N_1}; \qquad \frac{P_1}{P_2} \approx 1$$

Thus a-c power can be generated at any convenient voltage and the voltage can be increased and the current decreased without a serious loss of power by means of a transformer.   It can then be transmitted at this high voltage and low current to a point near where it is to be used, and then the voltage reduced and the current correspondingly increased by means of a second transformer.   Since the power loss in the transmission line varies as the square of the current for a given size of wire, the use of transmission voltages of 100 kilovolts or more leads to high transmission efficiencies.   The efficient reciprocal transformation of alternating current and voltage by means of this device is perhaps the most important reason for the nearly universal use of alternating currents.

2. An induced emf is produced in a coil of wire by relative motion of the coil and a magnet (Chap. XII).   In particular a stationary coil of wire and a rotatable magnet can be so arranged that a sinusoidal alternating emf is produced in the coil when the magnet is rotated at constant angular velocity by a mechanism such as a steam turbine.   Such *turbogenerators* are the most common sources of power.

3. Rotating a-c machines such as a-c generators or alternators and a-c motors are simpler to construct, more efficient, and easier to maintain than the corresponding d-c machines.

5.03. General Definitions.—The following definitions, some of which were given in Sec. 4.04, are stated in terms of current, but they are equally applicable to emfs, potential differences, or any other quantity that varies with time in the manner specified.

Alternating current is the name usually applied to a current that varies periodically with time and whose average value for one period is zero.   This means that $i = f(t)$ is an alternating current if there is a time interval T such that

$$(5.02) \quad f(t) = f(t + T) = f(t + 2T) = \cdots = f(t + nT)$$
$$\text{for } 0 \le t \le T$$

and

(5.03) $$\frac{1}{T} \int_{n\tau}^{(n+1)\tau} f(t)dt = 0$$

where n is any real number. The time interval T is called the *period* of the alternating current, and it is usually convenient to measure it in seconds. Since an alternating current traverses a complete cycle in T seconds, the *frequency*, or number of cycles per second, of the alternating current is

(5.04) $$f = \frac{1}{T}$$ cycles per second

According to the definition of an alternating current given above, it is not necessary for the positive and negative half cycles to be symmetrical, *i.e.*, for $f(t)$ to be equal to $-f\left(t + \dfrac{T}{2}\right)$. However, in most practical cases alternating currents are assumed to be symmetrical, and the methods of calculation described in this and subsequent chapters are based upon this assumption. Alternating-current circuits in which the currents are not symmetrical usually contain circuit parameters that cannot be represented by constants (such as R, L, C as they are specified in Chap. IV). Therefore an analytical solution is difficult (see Chap. IX).

The steady-state currents and voltages of a-c power circuits are usually so nearly sinusoidal that negligible error is introduced by assuming that they are. In those cases (particularly in the field of communications) in which the currents are not sinusoidal, the wave forms can often be expressed with good approximation as the sum of several sinusoidal currents of different frequencies. Examples of this procedure are presented in Chap. IX.

Throughout this chapter and in Chaps. VI, VII, and VIII it is assumed that the alternating quantities that are discussed are of the form representable by

(5.05) $$i = I_m \cos(\omega t - \theta)$$

where $\omega = 2\pi/T = 2\pi f$; ω is variously called the *angular velocity*, the *frequency constant*, or the *periodicity* (radians per second); T is the *period* (seconds); and f is the *frequency* (cycles per second). Two such sine waves are shown in Fig. 5.01 for two currents.

(5.06) $i_1 = I_{m1} \cos (\omega t - \theta_1)$
(5.07) $i_2 = I_{m2} \cos (\omega t - \theta_2)$

Note that the *phase angle** θ with respect to the abscissa for $t = 0$ corresponds to the horizontal distance from the origin
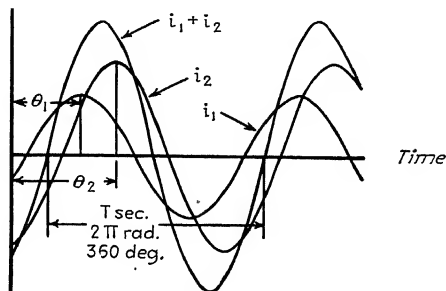
Fig. 5.01.

of the axes to the point where the wave first passes through its first *maximum* value.   The *difference in phase* between i_1 and i_2 is in this case $\theta_2 - \theta_1$.   It is said that i_2 *lags* i_1 or that i_1 *leads* i_2 by an angle $\theta_2 - \theta_1$.   Note that ωt and θ are to be expressed in radians or degrees when they are terms of the argument of the sine, but that they correspond to time intervals t and θ/ω if the axis of abscissas of the graph is marked in seconds.   The sum of the two alternating currents i_1 and i_2 of the same frequency is a sine wave of the same frequency; the ordinate of the curve marked $(i_1 + i_2)$ of Fig. 5.01 is, for any time t, the sum of the ordinates of the curves marked i_1 and i_2 at that instant.   The amplitude and the phase angle with respect to $t = 0$ of the sum are functions of the amplitudes and phase angles of the components.   These results can be demonstrated analytically (see Sec. 5.04) or graphically.

Simple methods of solving steady-state problems in a-c-circuit theory are based upon the choice of a means for specifying alternating currents and voltages unambiguously in terms less complicated than the time functions described above.   In other words, it is necessary to find a way to express, as a number of

* Note that, throughout this book, the symbol θ for a phase angle represents the absolute value of the angle in radians or degrees, *i.e.*, the symbol θ is a positive number.

amperes, a current that varies sinusoidally with time. The choice is determined by two facts already noted: (1) the rate at which (joule) heat is evolved at a particular instant in a resistor R is Ri^2, in which i is the current in the resistance at that instant, and (2) the instantaneous power input to a circuit is equal to the product of the instantaneous voltage drop across the terminals of the circuit and the instantaneous current through it, $p = vi$.

If an alternating current $I_m \cos (\omega t - \theta)$ flows through a resistance R, the rate of dissipation of heat (Ri^2) varies cyclically with time, but the average rate of dissipation P_{av} of heat throughout an integral number of cycles is

$$(5.08) \qquad \frac{1}{T} \int_0^T Ri^2 \, dt = \frac{RI_m^2}{T} \int_0^T \cos^2 (\omega t - \theta) \, dt$$

$$(5.09) \qquad P_{av} = R \frac{I_m^2}{2} \qquad\qquad \text{watts}$$

This result is often more important from the practical point of view than the function $RI_m^2 \cos^2 (\omega t - \theta)$. For example, the cyclic oscillation of the instantaneous power and therefore of the temperature of an electric toaster is usually of no interest, but the average power input to the toaster is a number that determines how fast the toast will burn. Similarly, if an a-c motor runs at constant speed to supply a constant load, the important practical problem in most cases is to find the average power input to the motor, rather than an expression of the instantaneous variation of the power input.* Thus if the difference of potential across the terminals of the motor is $v = V_m \cos (\omega t - \theta_v)$ and the current through the motor is $i = I_m \cos (\omega t - \theta_i)$, the average power input P_{av} is

$$(5.10) \quad \frac{1}{T} \int_0^T vi \, dt = \frac{V_m I_m}{T} \int_0^T \cos (\omega t - \theta_v) \cos (\omega t - \theta_i) \, dt$$

$$(5.11) \qquad\qquad P_{av} = \frac{V_m I_m}{2} \cos (\theta_v - \theta_i) \qquad\qquad \text{watts}$$

The current through a resistor is directly proportional to the voltage between its terminals at each instant ($v = Ri$). In this

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* Note, however, that an experimental investigation of instantaneous conditions may be required in the study of some problems, such as a study of vibrations in the parts of the motor or in the device which it is driving.

case $\theta_v = \theta_i$, and Eq. (5.11) reduces to $P_{av} = \dfrac{V_m I_m}{2}$. This result, together with Eqs. (5.09) and (5.11) imply that the *effective* value of a sinusoidal alternating quantity is $\dfrac{I_m}{\sqrt{2}}$. The *effective* or *root-mean-square (rms)* value I of a *sinusoidal* alternating quantity is defined as

$$(5.12) \qquad I = \sqrt{\tfrac{1}{T} \int_0^{} I_m^2 \cos^2 (\omega t - \theta)\, dt} = \frac{I_m}{\sqrt{2}} \text{ rms amperes}$$

Thus, if P is the average power input, I the rms current, V the rms voltage, and $\theta \equiv \theta_v - \theta_i$, then (5.09) and (5.11) can be rewritten

(5.13)   watts
(5.14) $P = VI \cos \theta$ watts

If the voltage and current are *in phase* ($\theta_v - \theta_i = 0$), the average power is $P = VI$ watts; if their phase difference is $\pi/2$, the average power is zero;* if their phase difference is π, the average power is $-VI$ watts. The minus sign is interpreted as meaning that the circuit is *delivering* power and not absorbing or transforming it.

The ratio of the average power input (to an a-c circuit) to the product of the rms voltage and the rms current is called the *power factor*. For sinusoidal voltages and currents it is equal to the cosine of the difference in phase between the voltage and current, according to Eq. (5.11). The power factor is conventionally expressed as a number from 0 to $+1$ or as a percentage equal to 100 cos θ where θ may have values in the first quadrant ($0 < \theta < \pi/2$) or in the fourth quadrant ($-\pi/2 < \theta < 0$). If the circuit in question has what appears to be a negative power factor (θ in the second or third quadrant), it is conventional to treat it as a source of energy with power input $-P$ and so to obtain a positive power factor. If the current leads the voltage the circuit is said to have a *leading* power factor, while if the current lags the voltage, the power factor is said to be *lagging*. These points are discussed further in Chap. VI.

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* A circumstance that can be approached closely, but never reached, in practice.

The specification of an alternating quantity by stating its rms value, frequency, and phase angle with respect to a zero chosen for convenience is discussed above from a theoretical point of view. Many experimental facts make this specification useful. For example, the mechanisms of many a-c voltmeters and ammeters[1] are such that the instantaneous torques tending to rotate the pointers of the instruments are proportional to the square of the instantaneous currents through them. Since the rotatable part of such a meter has usually too great an inertia to follow these variations of i^2 throughout the cycle, the pointer comes to rest at a point where the torque of the restraining spring of the instrument and the *average torque* produced by the current through the instrument are equal and opposite. The spring torque is usually proportional to the deflection and the second torque is proportional to the average value of the square of the current. *The deflection is therefore proportional to the square of the rms current* and the instrument scale is marked in rms values.

The essential fact is that the *effective* magnitude of an alternating current is specified in such a manner that its heating effect in a resistance is equal to that produced by a direct current of equal numerical value. Thus 10 (rms) amperes, 60 cycle, alternating current and 10 amperes direct current produce equal joule heating effects.

Although the average value of an alternating current for one cycle is zero, the average value for a complete half cycle is different from zero. This is often called the *average value* I_{av}. According to the notation described in this chapter the *form factor* and the *crest factor* are defined

$$(5.15) \quad \text{Form factor} = \frac{\ }{I_{av}} \qquad \text{and} \qquad \text{crest factor} =$$

The student should calculate the average value, the effective value, the form factor, and the crest factor of an alternating quantity having an amplitude I_m and a half-cycle consisting of each of the following: a sine wave from 0 to π, a rectangle, and a triangle.

5.04. The Addition of Sine Waves of the Same Frequency.—It was noted in Sec. 5.03 that the sum of two sine waves having the same frequency is a sine wave also. The analytical process

for proving this is straightforward.   The result is

(5.16) $\sqrt{2}\,I_1 \cos (\omega t - \theta_1) + \sqrt{2}\,I_2 \cos (\omega t - \theta_2)$

$\cos (\omega t -$

in which I_1, I_2, and I are rms currents and

(5.17) I $2I_1 l$ $\cos (\theta_2 -$

(5.18) $\sin$ $I_2 \sin \theta_2$

                   $\cos$ $I_2 \cos$

If the currents are in phase ($\theta_2 = \theta_1$), the resultant current has the rms value $I = I_1 + I_2$; *otherwise the sum of the two currents has an rms value less than the sum of their rms values.*   Although
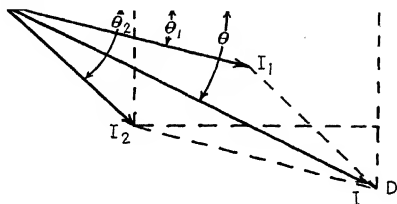
FIG. 5.02.

the algebraic sum of the *instantaneous* currents approaching a junction point in a circuit at any instant is zero if the charge at the junction is not changing, *the rms values of sinusoidal currents do not satisfy Kirchhoff's first law.*

Alternating quantities of sinusoidal form can be conveniently represented graphically by lines on a *polar diagram.**   Such a diagram is shown in Fig. 5.02.   Assume that all radial lines from the origin O rotate about O with constant angular velocity ω radians per second.   Let the line OP represent time $t = 0$; it will therefore represent also $t = T$, $t = 2T$, $t = 3T, \cdots$.   An alternating quantity such as

(5.05a) $i_1 = \sqrt{2}\,I_1 \cos (\omega t - \theta)$

is represented on the diagram by a radial line from O having a length proportional to I_1 and an angle with respect to OP of

* Such a diagram is commonly called a vector diagram.   Although the authors prefer to use the term *vector* for such quantities as those discussed in Chap. X., *i.e.*, space vectors, the terms *vector diagram* and *vector* for the diagrams and lines described in this and following chapters should be carefully noted by students; they have been justified by common usage.

$-\theta$.* This line represents in magnitude the rms value I_1, and in phase the instantaneous values of i_1 at the instants $t = 0$, $t = T$, $t = 2T$, $t = 3T$, $\cdots$. Since the phase relationships remain constant if the frequency of all quantities represented by the diagram is constant, the concept of the rotating lines may be dismissed as irrelevant. The current i_2 of amplitude and phase different from those of i_1 can be similarly drawn as shown in Fig. 5.02. Next draw a line from the end of the line representing I_1 parallel to the line representing I_2, and a corresponding line parallel to the line representing I_1 through the end of the I_2 line. These lines intersect at D. Draw the line OD.

Inspection of the completed diagram shows that

$$\overline{I_2^2 + 2\overline{I_1}\overline{I_2}} \cos (\theta_2 - \tag{5.17}$$

$$\tan \theta = \frac{I_1 \sin \theta_1 + I_2 \sin \theta_2}{I_1 \cos \theta_1 + I_2 \cos \theta_2} \tag{5.18}$$

These are precisely the results obtained analytically for the sum of two sinusoidal quantities [see Eqs. (5.17) and (5.18) in the first paragraph of this section]. Thus the scheme described above is a method for representing sinusoidal quantities, and for adding† them, graphically.

Figure 5.03 is a polar diagram representing V‡, the rms input voltage to an alternating-current circuit, and I‡ the rms current flowing in the circuit; V having been chosen as the zero reference ($\theta = 0$). The line $\overline{OA} = I \cos \theta$ is called the *component* of I that is in phase with V. Note that $V(I \cos \theta)$ is the average power input. The line $\overline{AB} = I \sin \theta$ is called the component of I that is in *quadrature* (*i.e.*, 90 degrees out of phase) with V. It is apparent that $V(I \sin \theta)$ has a definite magnitude but that it has nothing to do with the average power input (see also Sec.

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* A current $i' = \sqrt{2}\, I' \cos (\omega t + \theta')$ would accordingly be plotted at an angle $+\theta'$ from OP. Thus the conventions chosen for representing currents and voltages on a polar diagram are the same as those universally used for trigonometry: *positive angles are measured counterclockwise from a horizontal line extending to the right from the origin.*

† The process used to obtain a line for I is the process used for adding two coplanar vectors as the student will recall from his study of mechanics.

‡ Note that, according to the conventions adopted in this book, the instantaneous voltage and current are: $v = \sqrt{2}\, V \cos \omega t$;

$$i = \sqrt{2}\, I \cos (\omega t + \theta).$$

6.04).   The product of the instantaneous values of i and v is

(5.19) $vi = 2VI \cos (\omega t + \theta_i) \cos \omega t$

$\qquad = 2VI(\cos^2 \omega t \cos \theta_i - \sin \omega t \cos \omega t \sin \theta_i)$

Thus the graphical analysis of Fig. 5.03 shows that the first term of the right-hand member of Eq. (5.19) corresponds to the quantity $V(I \cos \theta)$ while the second term of Eq. (5.19) corre-
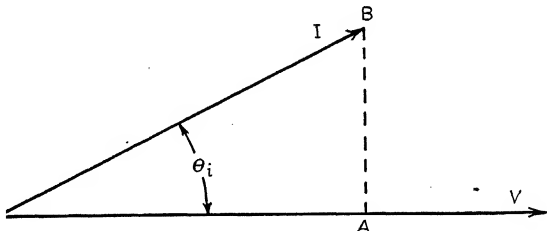
Fig. 5.03.

sponds to the quantity $V(I \sin \theta)$.   Further discussion of this point is postponed until the cases of a leading current such as that represented by Fig. 5.03 are analyzed.

Since a graphical method of solving problems is in general tedious and likely to be inaccurate, an analytical method is usually to be preferred.   An analytical method of solving a-c-network problems is presented in Chap. VI.   The graphical method discussed in this section and in the next section is the basis of the analytical method.

5.05. Reactance and Impedance ; Susceptance and Admittance. In this section the graphical method of representing alternating quantities described above and the steady-state solutions of simple a-c circuits derived in Chap. IV are combined for the purpose of (1) finding quantities associated with resistors, inductors, and condensers that can be measured and used in the (2) solving of these simple steady-state problems by a simpler method than that used in Chap. IV.   Throughout this section the current in the circuit element of a mesh under consideration will be chosen as the reference quantity, *i.e.*, its phase angle is assumed to be zero.

When a sinusoidal emf is connected to the terminals of a resistor, the current and emf are in phase

(5.20) $\qquad\qquad \sqrt{2}\, E \cos \omega t = \sqrt{2}\, RI \cos \omega t$

so that, *in this particular case,*

$$(5.21) \qquad\qquad E = RI \qquad\qquad \text{volts}$$

Thus the rms current and voltage in an a-c circuit containing a sinusoidal emf and a resistance load are related in the same manner as the d-c current and voltage in a corresponding d-c circuit. The resistor dissipates heat at the same rate as in a d-c circuit *if the d-c* and *a-c resistances of the resistor are equal.* However, if the frequency is sufficiently great, the a-c resistance of a conductor is greater than its d-c resistance. At high frequencies the current stream lines are more concentrated near the periphery
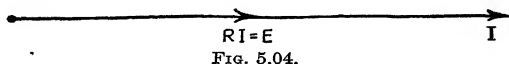
$$R\,I = E$$
Fig. 5.04.

of a cylindrical conductor than near the center. This is called the *skin effect* and it is discussed in Chap. XIII. At frequencies less than a few hundred cycles per second the effect is usually negligible from the practical point of view. In any case the effective resistance of a conductor can be measured by measuring the rate of evolution of heat P, while a known steady-state current I flows in it; the resistance is

$$(5.22) \qquad\qquad R = \qquad\qquad \text{ohms}$$

The simple circuit discussed in this paragraph can be represented by the polar diagram shown in Fig. 5.04, which is constructed according to the method described in Sec. 5.04.

When a sinusoidal emf is connected to a resistor and condenser in series, the current and emf are *not* in phase. According to Eq. (4.34), the steady-state current is (for $\phi_1 = 0$)

$$(5.23) \qquad i = \frac{-\sqrt{2}\,E}{\sqrt{R^2 + \dfrac{1}{\omega^2 C^2}}} \sin\left[\omega t - \tan^{-1}(\omega RC)\right] \qquad \text{amperes}$$

or

$$(5.24) \qquad i = \frac{\sqrt{2}\,E}{\sqrt{R^2 + \dfrac{1}{\omega^2 C^2}}} \cos\left[\omega t + \frac{\pi}{2} - \tan^{-1}(\omega RC)\right]$$

Thus the polar diagram for Eq. (5.24) is that shown at the left, Fig. 5.05. Since the current is chosen as reference (zero angle on the polar diagram), the figure is so redrawn on the right, Fig. 5.05. The rms voltage drop through the resistance (RI) is in
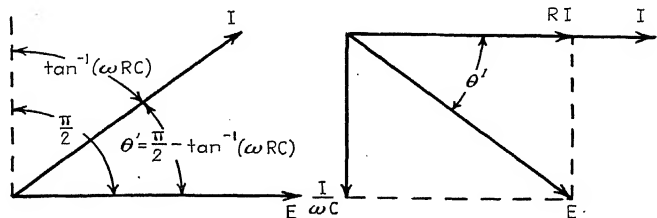
FIG. 5.05.

phase with the current; the magnitude and phase angle of the *voltage* across the resistor *and* condenser are

$$(5.25) \qquad E = I \sqrt{R^2 + \frac{1}{\omega^2 C^2}} \qquad \text{rms volts}$$

$$(5.26) \qquad \theta' = \tan^{-1}\left(\frac{1}{\omega R C}\right) \qquad \text{radians or degrees}$$

so that

$$(5.26a) \qquad e = \sqrt{2}\sqrt{R^2 + \frac{1}{\omega^2 C^2}}\, I \cos(\omega t - \theta') \qquad \text{volts}$$

The voltage drop through the capacitance has a magnitude $I/\omega C$ and it lags the current by $\pi/2$ radians. Thus the voltage E across the resistor and condenser is the resultant of RI and $I/\omega C$ calculated according to Eq. (5.17), and the phase of E with respect to I can be calculated from Eq. (5.18).

The ratio of the rms voltage to the rms current that it causes to flow in a circuit is called the *impedance Z* of the circuit. Thus for a resistor and condenser in series the impedance is, from Eq. (5.25)

$$(5.27) \qquad Z = \frac{E}{I} = \sqrt{R^2 + \frac{1}{\omega^2 C^2}} \qquad \text{ohms}$$

The absolute magnitude of the ratio of the rms voltage across the terminals of a capacitance to the rms current through it is

commonly called the *capacitative reactance* of the condenser while the *effective reactance* X_c is defined as

$$(5.28) \qquad\qquad X_c = -\frac{1}{\omega C} \qquad\qquad \text{ohms}$$

The choice of the negative sign is not determined by the analysis given above. In general it is convenient—and this point will be clarified later in this section—to define the effective reactance of a portion of a circuit as negative if the voltage drop through it lags behind the current and as positive if the voltage leads the current.

Impedance and reactance are expressed in ohms. It is sometimes convenient to calculate and to use the reciprocals of the
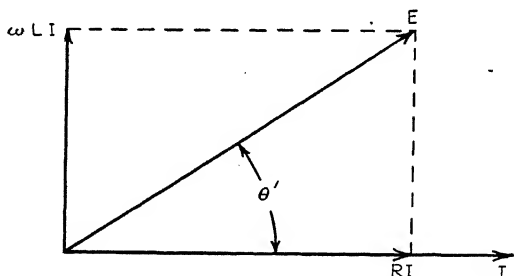
Fig. 5.06.

impedance and reactance. These are called respectively the *admittance Y* and the *susceptance B*, and they are expressed in mhos. The emphasis in this book is placed on the quantities, impedance and reactance. Admittance and susceptance can be used advantageously in the calculations related to parallel circuits (Sec. 5.06).

Following the same procedure as that described above for Eq. (4.34), the polar diagram for the steady-state component of Eq. (4.41) is found to be that shown in Fig. 5.06 (for $\phi_1 = 0$)· Thus corresponding to Eqs. (5.25) to (5.28) for a circuit comprising a resistor and condenser, there are the following relations and definitions for a circuit comprising an inductance and resistance:

$$(5.29) \qquad\qquad E = I \sqrt{R^2 + \omega^2 L^2} \qquad\qquad \text{volts}$$

$$(5.30) \qquad\qquad \tan^{-1} \frac{\omega L}{R} \qquad\qquad \text{radians or degrees}$$

$$(5.31) \qquad\qquad Z = \sqrt{} \qquad\qquad \text{ohms}$$

$$(5.32) \qquad\qquad X_L = +\omega L \qquad\qquad \text{ohms}$$

so that

$$(5.32a) \qquad e = \sqrt{2}\,\sqrt{R^2 + \omega^2 L^2}\, I \cos(\omega t + \theta') \qquad\qquad \text{volts}$$

Note that in this case the voltage leads the current, and the effective reactance is therefore taken as $+\omega L$.

When the procedure is followed once more for the steady-state component of Eq. (4.61), the results are

$$(5.33) \qquad\qquad E = I \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \qquad\qquad \text{volts}$$

$$(5.34) \qquad\qquad \theta' = \tan^{-1}\left[\frac{\omega L - \dfrac{1}{\omega C}}{R}\right] \qquad\qquad \text{radians or degrees}$$

$$(5.35) \qquad\qquad Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \qquad\qquad \text{ohms}$$

$$(5.36) \qquad\qquad X = \omega L - \frac{1}{\omega C} \qquad\qquad \text{ohms}$$

so that

$$(5.36a) \quad e = \sqrt{2}\,\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\, I \cos(\omega t + \theta') \qquad\qquad \text{volts}$$
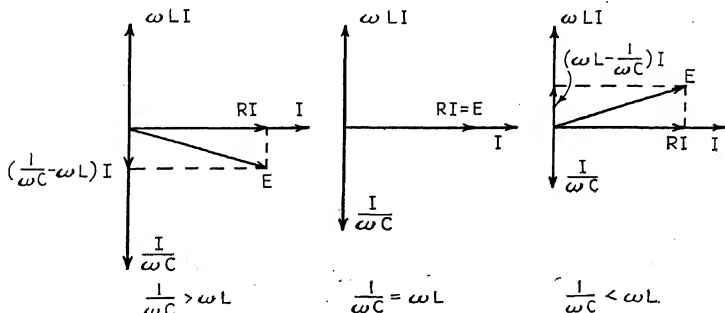
Fig. 5.07.

The polar diagrams for three values of X are shown in Fig. 5.07. When $\omega L = 1/\omega C$, the polar diagram shows that the voltage and

current are in phase, as shown analytically in Chap. IV [Eq. (4.63)]. The circuit is said to be *resonant* when the reactance is zero.

5.06. Impedances in Series and in Parallel.—The equivalent impedance of several impedances in series can be calculated from their individual resistances and reactances. If the same current I flows through each of a series of impedances $Z_1, Z_2, \cdots, Z_n$, the voltage drop attributed to the resistance of each impedance is *in phase* with the current. Therefore the resistances can be added arithmetically, the *effective* reactances can be added algebraically, and the equivalent impedance can be calculated from

(5.37)
$$Z = \sqrt{(R_1 + R_2 + \cdots} \qquad \overline{) + (X_1 + X_2 +} \qquad \text{ohms}$$

The process is illustrated in Fig. 5.08 for three impedances, two having positive effective reactance and one having negative
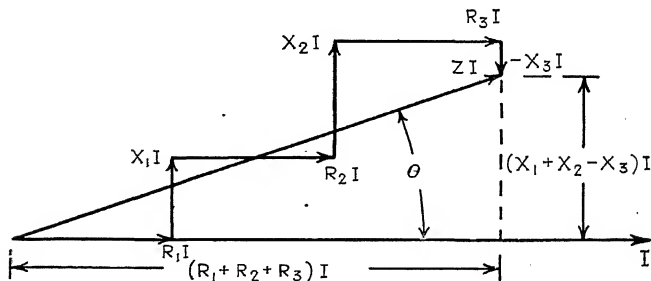
Fig. 5.08.

effective reactance. Note that the net voltage drop ZI leads the current by an angle θ

(5.38)
$$\theta = \tan^{-1} \frac{X_1 \qquad X_3}{R_1 + R_2 + R_3}$$

If the voltage drop through a series of impedances is V and the current through it is I, the power input to the series is

(5.39)
$$P = VI \cos \theta \qquad \text{watts}$$

or

(5.40)
$$P = \frac{V^2}{Z} \qquad \text{watts}$$

From Fig. 5.08,

(5.41)
$$\cos \theta = \frac{R}{\sqrt{R^2 + X^2}}$$

so that

(5.42)
$$P = \frac{R}{R^2 + X^2} V^2 \qquad \text{watts}$$

The quantity $\dfrac{R}{R^2 + X^2}$ is called the *effective conductance G* of the impedance Z.   The effective admittance Y is the reciprocal of Z. Therefore since

(5.43)
$$Y^2 = G^2 + B^2$$

(5.44)
$$B^2 = \frac{1}{R^2 + X^2} - \frac{R^2}{(R^2 + X^2)^2}$$

and the *effective susceptance B* is

(5.45)
$$B = \frac{-X}{R^2 + X^2} \qquad \text{mhos}$$

Note that the sign $(-)$ of the effective susceptance corresponds to the choice of signs for the effective reactances discussed above. It follows from the definitions given above that

(5.46)
$$I = YE \qquad \text{amperes}$$

in which

(5.47)
$$Y = \sqrt{G^2 + B^2} \qquad \text{mhos}$$

If the same voltage E is applied to a number of impedances in parallel, the component of the current in each impedance attributable to its effective conductance is *in phase* with E while the component attributable to its effective susceptance is ± 90 degrees out of phase with E.   Therefore the effective conductances can be added arithmetically, the effective susceptances can be added algebraically, and the equivalent admittance can be calculated from

(5.48)
$$Y = \frac{1}{Z}$$
$$= \sqrt{(G_1 + G_2 + \cdots + G_n)^2 + (B_1 + B_2 + \cdots + B_n)^2} \qquad \text{mhos}$$

The process is illustrated in Fig. 5.09 for three admittances.

This concludes the discussion which has led to (1) definitions of the quantities that are used in a-c circuit theories and (2) a means for solving single-mesh a-c circuits without resorting to the differential equations representing them. The ground-work for the study of steady-state conditions in a-c circuits is
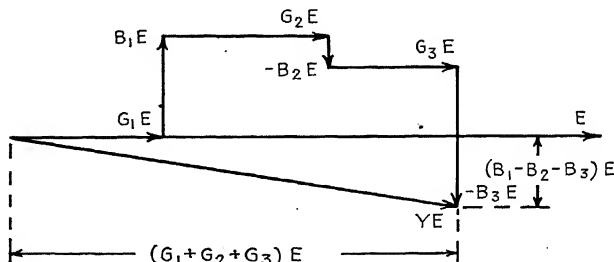
Fig. 5.09.

complete. Subsequent chapters are devoted to the application of this groundwork to the methods used for solving practical engineering problems.

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Problems

5.01. Calculate I, I_1, I_2, and represent them and the input voltage on a polar diagram; see figure Prob. 5.01.
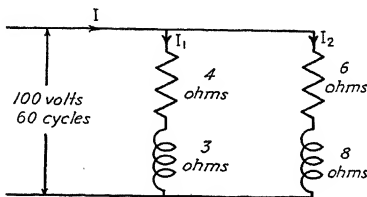
PROB. 5.01.

5.02. Calculate the frequency f for maximum current, and the current at f cycles per sec. and at 60 cycles per sec. in a series circuit for which $E = 200$ volts; $R = 10$ ohms; $L = 0.1$ henry; $C = 100$ μf. Calculate the voltage across each circuit element for each of the two frequencies.

5.03. Draw a polar diagram for an RL circuit in which $E = 220$ volts; $I = 15.2$ amp.; $P = 2{,}126$ watts; $f = 60$ cycles per sec. Calculate R and L.

5.04. Find the average, maximum, and minimum power input to a series circuit for which $I = 3.2$ amp.; $R = 40$ ohms; $L = 0.12$ henry; $C = 40\mu f$; $f = 60$ cycles per sec.

5.05. A resistance R, inductance L, and capacitance C are connected in series with an emf E (rms volts). As the frequency of the emf is varied, it is found that maximum power input to the RLC circuit is delivered at 60 cycles; the power input is then 3,851 watts and the current is 35 amp. (rms). At 25 cycles the power input is 314.4 watts. Calculate R, L, C, and E.

5.06. Calculate the rms value and the average value of a current of which one cycle is of the form

$$i = 0.1\varepsilon^{-2t} \qquad 0 \le t \le 0.05 \text{ sec.}$$

and

$$0 \qquad 0.05 \le t \le 0.10 \text{ sec.}$$

What is the rms value of a current i' which differs from i in that the current for $0.05 \le t \le 0.10$ sec. is $i' = -0.1\varepsilon^{-2t}$?

5.07. If the voltages between wires for each of two circuits are

$$200 \sin \qquad + \tfrac{..}{4} \qquad \text{volts}$$

and

$$v_2 = 300 \sin \qquad - \tfrac{5\pi}{\cdot}) \qquad \text{volts}$$

what readings can be obtained by connecting a wire from one wire of one circuit to one wire of the second circuit and by connecting a voltmeter to the remaining (free) wires?

5.08. Six wires are connected at a point A. The currents approaching it in 5 of the wires are

$$2.828 \sin \left(2\pi 60t + \frac{\pi}{2} \right) \quad \text{amp.}$$

$$i_2 = 2.828 \sin \left(2\pi 120t \right) \quad \text{amp.}$$
$$i_3 = 2.828 \cos \left(2\pi 320t \right) \quad \text{amp.}$$

$$i_4 = 2.828 \sin \left(2\pi 60t - \frac{\pi}{2} \right） \quad \text{amp.}$$

$$i_5 = 1.000 \quad \text{amp.}$$

Two ammeters are connected in series with the sixth wire. One indicates rms current; the other, average current. Calculate their readings.

5.09. Calculate the components of Z, so chosen that the input voltage and the input current are in phase.
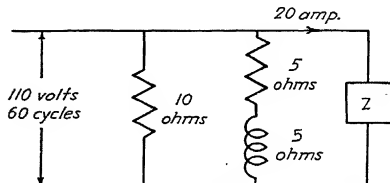
PROB. 5.09.

5.10. Calculate the smallest current in Z (Problem 5.09) that would be required to obtain unity power factor for the whole circuit.

5.11. Two coils, having resistances of 5 ohms and 10 ohms and inductances of 0.02 henry and 0.01 henry respectively, are connected in parallel to a source of 60-cycle alternating current. Calculate for a single impedance equivalent to this parallel circuit the resistance, reactance, impedance, effective susceptance, and effective conductance.

5.12. The impedance of a leaky condenser is 100 ohms at 100 cycles and 11 ohms at 1,000 cycles. Describe a circuit of two elements that is equivalent to the condenser, and calculate the power input to the condenser when the voltage between its terminals is 100 volts.

5.13. If $e = \sqrt{2}\, E \sin \omega t$ and $R_1 = R$, calculate C in terms of L, R, and ω for the condition of maximum power dissipation in R_1.
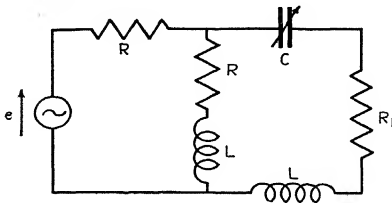
PROB. 5.13.

5.14. A well-designed resistor has distributed inductance and capacitance that are negligibly small except for very high frequencies (> 1 megacycle per

sec.). Derive an approximately equivalent circuit for the resistor at high frequencies and calculate the effective resistance and the effective reactance of the approximately equivalent circuit.

5.15. Two coils, having resistances 15 ohms and 7 ohms and inductances L_1 and L_2, are connected in series to a source of 220 volts, 60 cycles. The power input to the circuit is 550 watts and the voltages across the coils are equal. Calculate the reactance of each coil.

5.16. A coil having a resistance of 10 ohms and an inductance of 0.05 henry is connected in series with a condenser and a source of 220 volts, 25 cycles. Calculate the capacitance of the condenser for which the power factor of the circuit is 0.95, leading. What are the maximum, minimum, and average power inputs to the circuit?

5.17. A single-mesh circuit is connected to a variable-frequency source whose terminal voltage is 115 volts. The current at 25 cycles is 8 amp. As the frequency is increased, the current increases to a maximum value of 24 amp. at 55 cycles; the current then decreases for further increase in frequency. What are the elements of the circuit? What is the power factor at 25 cycles?

5.18. Can an *RLC* series circuit be devised so that the rms voltages across each element are equal when the circuit is connected to a source of alternating current?

THE USE OF COMPLEX NUMBERS IN SOLVING ALTERNATING-CURRENT NETWORKS

6.01. Introduction.—It was noted in the last chapter that coexisting alternating quantities of frequency f can be represented by radial lines that are assumed to rotate with constant angular velocity $2\pi f$ radians per second. The relative positions of the lines in such a polar diagram remain constant if the lines represent *steady-state* alternating quantities. Thus a statement of the frequency f and a *fixed* polar diagram for one instant comprise a complete specification of the *steady-state* problem. The instant for which the diagram is given can be chosen arbitrarily. It is convenient to choose the instant for which one of the quantities has a phase angle of zero as a reference and so to draw a horizontal reference line representing this quantity. The line representing each of the other quantities is then drawn at an angle to the reference line equal to the difference between the phase angles of the quantities that the lines represent. These methods and a graphical means of adding component voltage drops to obtain the resultant are described in Chap. V.

It is the purpose of this chapter to present a method whereby the graphical methods described above can be expressed analytically. This leads to a method of calculating the magnitudes and phase angles of the steady-state currents in an a-c network in terms of the emf's and the impedances of the network. In this method a line on a polar diagram is represented analytically by a *complex number*. The representation of a current, voltage, impedance, or other quantity by a complex number is commonly called *vector,* * *symbolic*, or *complex-number notation*.

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* Note here again that this is not a space-vector notation. Complex-number representations of current and voltage and impedance are commonly called *vector current, vector voltage,* and *vector impedance*. These latter terms will accordingly be used throughout this chapter and those following.

The relevant properties of complex numbers are reviewed in the next section.

6.02. Certain Properties of Complex Numbers.—A number consists in general of a *real number* and an *imaginary number* that are combined in a prescribed manner. Real numbers comprise zero, positive and negative integers, positive and negative rational fractions, and positive and negative irrational numbers. Imaginary numbers comprise real numbers multiplied by the *unit imaginary*, which is defined as the *positive square root of minus one*. To designate the unit imaginary the symbol used is i in most mathematical literature but j in the literature of physics and electrical engineering, because in the latter i is used as a symbol for electric current.

Real numbers can be represented by points on a straight line; pure imaginary numbers can be similarly represented. If two perpendicular lines are used to represent real and imaginary numbers, each point in the plane of these axes represents a particular *complex number* to which there corresponds a *real part* and an *imaginary* part, either of which may be zero. Rules for manipulating complex numbers can be derived from the concepts of the algebraic properties of the parts and of the square root of minus one, and from the geometric properties of lines on the plane containing the perpendicular axes. Those results of this procedure that are relevant to its application to the calculations of electric-circuit theory are presented below.

Complex numbers are represented by points in the plane of an *Argand diagram,* described above and illustrated in Fig. 6.01. The vertical axis, called the *axis of imaginaries,* includes points representative of imaginary numbers of the form $j\beta$, in which β is any real number including zero; real numbers α are represented by points on the horizontal *axis of reals.* A complex number $\underset{\cdot}{A}$ is defined in terms of its real part α and its imaginary part* $j\beta$, where β is a real number, *viz.*,

$$(6.01) \qquad\qquad \underset{\cdot}{A} = \alpha$$

where the dot is placed under the A to indicate that $\underset{\cdot}{A}$ is a complex number and the expression $\alpha + j\beta$ means: *find the point on the axis of reals corresponding to α and the point on the axis of*

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* Note that the real number β is commonly called "the imaginary part of the complex number $\underset{\cdot}{A}$."

imaginaries corresponding to β; draw lines perpendicular to the respective axes at these points; their point of intersection is the point representing the complex number $A = \alpha + j\beta$. The form of expressing a complex number A given above in Eq. (6.01) is usefully called the *rectangular form.* This rather complicated operation corresponding to an algebraic notation of the sum of a real and an imaginary number turns out to be extremely useful. For example, the sum of two complex numbers

$$A_2 = \alpha$$

corresponds to a complex number A represented by a point P having a coordinate on the axis of reals equal to $\alpha_1 + \alpha_2$ and a coordinate on the axis of imaginaries equal to $\beta_1 + \beta_2$ where the α's and the β's are added algebraically:

.(6.02) $A =$

This process of addition is illustrated in Fig. 6.01, in which the point P_1 corresponds to A_1, P_2 to A_2, and P to A.   Inspection
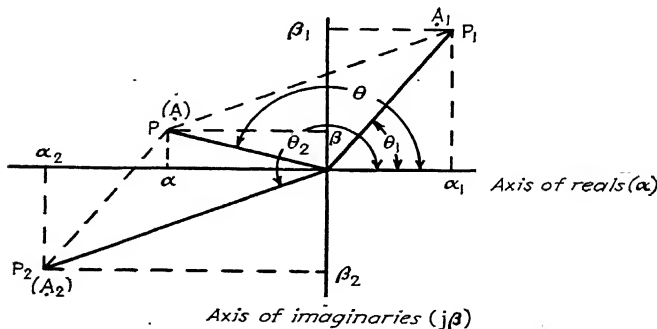
FIG. 6.01.

of this diagram shows that the process of adding A_1 to A_2 is similar to the process described in Chap. V for combining two lines in a polar diagram representing two sinusoidal currents.

A complex number can be represented analytically in several ways. For example, note that the distance r_1 from the origin (Fig. 6.01) to the point P_1 is $(\alpha_1^2 + \beta_1^2)^{\frac{1}{2}}$ and the angle between r_1

and the positive axis of reals is θ_1.   The point P corresponding to A_1 can be specified by writing

(6.01) $A_1 = \alpha_1 + j\beta_1$

or

(6.03) $A_1 = A_1\underline{/\theta_1};$ $A_1 \equiv +\sqrt{\alpha_1^2 + \beta_1^2};$ $\theta_1 \equiv \tan^{-1}\frac{\beta_1}{\alpha_1}$

where the positive real number represented by A_1 (without the dot) is called the *magnitude* or *modulus* of A_1, and θ_1 is called the *phase angle* or *argument* of A_1.   The symbols $A_1\underline{/\theta_1}$ are read, "A one phase theta one," and $A_1\underline{/\theta_1}$ is commonly called the *polar form* of the complex number A_1.

The expression $A_1\underline{/\theta_1}$ can be written in another manner.   It will be recalled that the Maclaurin expansion of ε^x is

(6.04) $\varepsilon^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

If the imaginary number $j\theta_1$ is substituted for x, and the right-hand member is divided into its real and imaginary parts there results

(6.05) $\varepsilon^{j\theta_1} = \left[1 - \frac{\theta_1^2}{2!} + \frac{\theta_1^4}{4!} - \frac{\theta_1^6}{6!} + \cdots\right]$
$$+ j\left[x - \frac{\theta_1^3}{3!} + \frac{\theta_1^5}{5!} - \frac{\theta_1^7}{7!} + \cdots\right]$$

because $j = \sqrt{-1}; j^2 = -1; j^3 = -j; j^4 = +1$.   The quantities in the parentheses of Eq. (6.05) are respectively the Maclaurin expansions of $\cos\theta_1$ and $\sin\theta_1$ so that

(6.06) $\varepsilon^{j\theta_1} = \cos\theta_1 + j\sin\theta_1$

It follows from Eq. (6.06) that

(6.07) $A_1\varepsilon^{j\theta_1} = A_1\cos\theta_1 + jA_1\sin\theta_1$

and since from Eq. (6.03)

$A_1 = \sqrt{\alpha_1^2 + \beta_1^2};$ $\cos\theta_1 = \frac{\alpha_1}{\sqrt{\alpha_1^2 + \beta_1^2}};$ $\sin\theta_1 = \frac{\beta_1}{\sqrt{\alpha_1^2 + \beta_1^2}}$

the complex number A_1 can also be written

(6.08) $A_1 = A_1\varepsilon^{j\theta_1}$

Thus $\underline{/\theta_1}$ and $\varepsilon^{j\theta_1}$ are equivalent from the analytical point of view.

When complex numbers are to be added or subtracted it is most convenient to express them in the rectangular form of Eq. (6.01). Complex numbers are most easily multiplied and divided if they are expressed in the polar form of Eq. (6.08) or Eq. (6.03). Thus the product of A_1 and A_2 is

$$(6.09) \quad A_1 A_2 = A_1 A_2 \varepsilon^{j(\theta_1 + \theta_2)} = A_1 A_2 [\cos (\theta_1 + \theta_2) + j \sin (\theta_1 + \theta_2)]$$

If this quantity is expanded and values of A_1, A_2, $\cos \theta_1$, $\cos \theta_2$, $\sin \theta_1$, $\sin \theta_2$ in terms of α_1, β_1, α_2, β_2, the result is

$$(6.10) \quad A_1 A_2 = \alpha_1 \alpha_2 - \beta_1 \beta_2 + j(\alpha_1 \beta_2 + \alpha_2 \beta_1)$$

This is precisely the result that is obtained if the operation $(\alpha_1 + j\beta_1)(\alpha_2 + j\beta_2)$ is carried out according to the rules of algebra, and -1 is substituted in the result for j^2.

Complex numbers are easily divided if they are expressed in the polar form of Eq. (6.08) or Eq. (6.03)

$$(6.11) \quad \frac{A_1}{A_2} = \frac{A_1}{A_2} \varepsilon^{j(\theta_1 - \theta_2)} = \frac{A_1}{A_2} [\cos (\theta_1 - \theta_2) + j \sin (\theta_1 - \theta_2)]$$

If this quantity is expressed in terms of α_1, β_1, α_2, and β_2, the result is

$$(6.12) \quad \frac{A_1}{A_2} = \frac{(\alpha_1 \alpha_2 + \beta_1 \beta_2) + j(\alpha_1 \beta_2 - \alpha_2 \beta_1)}{\alpha_2^2 + \beta_2^2}$$

This is a result that can also be obtained as follows:

$$(6.13) \quad \frac{A_1}{A_2} = \frac{\alpha_1 + j\beta_1}{\alpha_2 + j\beta_2} = \frac{(\alpha_1 + j\beta_1)(\alpha_2 - j\beta_2)}{(\alpha_2 + j\beta_2)(\alpha_2 - j\beta_2)}$$
$$= \frac{(\alpha_1 \alpha_2 + \beta_1 \beta_2) + j(\alpha_1 \beta_2 - \alpha_2 \beta_1)}{\alpha_2^2 + \beta_2^2}$$

The process used to obtain Eq. (6.13), *i.e.*, the multiplication of a complex numerator and a complex denominator by the denominator with the sign of the imaginary part changed, is called *rationalization*. Quantities of the form $(\alpha + j\beta)$ and $(\alpha - j\beta)$ are called *conjugate complex numbers*. Note in particular that the product of a complex number $\alpha + j\beta$ by its conjugate $\alpha - j\beta$ is a real number equal to the square of the modulus of either quantity.

$$(6.14) \quad (\alpha + j\beta)(\alpha - j\beta) = (\alpha^2 + \beta^2)^{\frac{1}{2}}(\alpha^2 +$$

The student should learn to perform the operations on complex numbers described above quickly and accurately. The mathematical principles underlying these operations are discussed in detail in the mathematical literature.[1] Trigonometric identities, the roots of complex, real, and imaginary numbers, and the fundamental theory of functions of a complex variable can be obtained by developing further the principles discussed in this section.

6.03. Complex–number or Vector Expressions of Current, Voltage, and Impedance.—It was shown in Chap. V that sinusoidal alternating quantities can be represented by lines on a
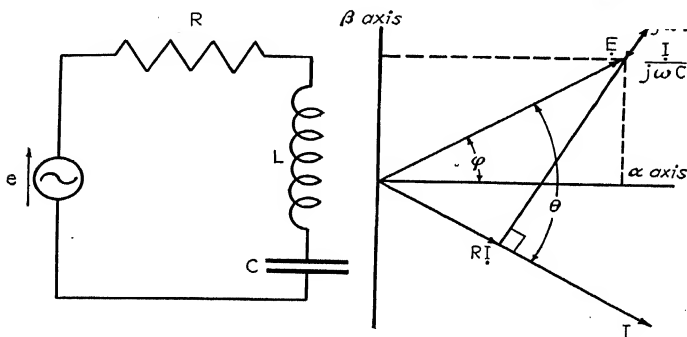
FIG. 6.02.

polar diagram. In turn these lines can be expressed in terms of complex numbers and all calculations made analytically in terms of them. The procedure is illustrated below by an analysis of the single-mesh circuit shown in Fig. 6.02, *left.* The corresponding polar diagram, in which horizontal distances are assumed to represent real numbers and vertical distances are assumed to represent imaginary numbers, is shown in Fig. 6.02, *right.* The impressed voltage E is assumed to have a phase angle ϕ with respect to the zero reference line. Using the notation described in Sec. 6.02, the *complex-number* or *vector* representation E of the voltage shown in the polar diagram of Fig. 6.02 is

(6.15) $= E_\alpha$

The voltage drop through R is in phase with the current whose

phase angle is as yet unknown

$$(6.16) \qquad\qquad V_R = RI \qquad\qquad \text{vector volts}$$

The voltage drop V_L through the inductance L *leads* the current by 90 degrees. According to the rules discussed in Sec. 6.02 a line in an Argand diagram represented by jI leads the line represented by I by 90 degrees. Therefore

$$(6.17) \qquad\qquad V_L = j\omega L I \qquad\qquad \text{vector volts}$$

Similarly, multiplication of a complex quantity I by $-j \left(= \dfrac{1}{j} \right)$ produces a complex quantity $-jI$ which *lags* I by 90 degrees. Therefore the voltage drop V_C through the capacitance C is

$$(6.18) \qquad\qquad V_C = -\frac{jI}{\omega C} \qquad\qquad \text{vector volts}$$

The sum of these three voltage drops is equal to the impressed emf E. Thus

$$(6.19) \qquad\qquad E = \left[R + j\left(\omega L - \frac{1}{\omega C} \right) \right] I \qquad\qquad \text{vector volts}$$

The complex quantity in the square brackets is chosen by definition as the *complex-number* or *vector* or *symbolic* expression of the impedance Z.

$$(6.20) \qquad\qquad Z = R + j\left(\omega L - \frac{1}{\omega C} \right) \qquad\qquad \text{vector ohms}$$

$$(6.21) \qquad Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C} \right)^2} \Bigg/ \tan^{-1} \frac{\omega L - \dfrac{1}{\omega C}}{R}$$

Therefore the particular problem illustrated in Fig. 6.02 can be solved analytically in terms of complex numbers as follows. The current I is

$$(6.22) \qquad\qquad I = \frac{E}{Z} = \frac{E\underline{/\phi}}{Z\underline{/\theta}} = \frac{E}{Z} \underline{/\phi - \theta} \qquad\qquad \text{vector amperes}$$

The rms current I is

$$(6.23) \qquad\qquad I = \frac{E}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C} \right)^2}} \qquad\qquad \text{rms amperes}$$

and the angle between the reference zero and I is $(\phi - \theta)$ in which ϕ is the phase angle of the voltage with respect to the chosen zero reference and, from Eq. (6.21),

$$\theta = \tan^{-1} \frac{\omega L - \dfrac{1}{\omega C}}{R} \qquad \text{radians or degrees}$$

According to the rules by which the polar diagram was constructed, the instantaneous value of the steady-state current is

$$(6.24) \qquad i = \frac{\sqrt{2}\, E}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}} \cos\,(\omega t - \theta + \phi) \qquad \text{amperes}$$

in accord with the particular solution obtained in Chap. IV for the differential equation of the circuit.

The method of expressing **steady-state** voltages and currents, and impedances, as complex numbers together with the relations among them have been derived from an analysis of polar diagrams representing these quantities, and the properties of complex numbers. The same result can be obtained directly from the differential equation representing the circuit. Furthermore, this derivation leads immediately to a means for calculating steady-state conditions in a-c circuits of more than one mesh. This approach to the results given by Eqs. (6.22), (6.23), and (6.24) for the circuit shown in Fig. 6.02, *left*, is therefore presented below in detail.

The differential equation representing the circuit of Fig. 6.02, *left* is

$$(6.25) \quad L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{i}{C} = \frac{de}{dt} = -\sqrt{2}\,\omega E \sin\,(\omega t + \phi)$$

The right-hand member can be written as the sum of two terms as follows

$$(6.26) \qquad -\sqrt{2}\,\omega E \sin\,(\omega t + \phi) = A\varepsilon^{j\omega t} + B\varepsilon^{-j\omega t}$$

in which

$$(6.27) \qquad A = -\frac{\omega E \varepsilon^{j\phi}}{\sqrt{2}\,j} \qquad \text{and} \qquad B = \frac{\omega E \varepsilon^{-j\phi}}{\sqrt{2}\,j}$$

Equations of the form (linear) of Eq. (6.25) have the property

that if two independent solutions i' and i'' satisfy the equation, then $i' + i''$ is also a solution.   Thus if two solutions,

$$i' \text{ corresponding to } A \varepsilon^{j\omega t}$$

and

$$i'' \text{ corresponding to } B \varepsilon^{j\omega t},$$

the sum* $i' + i''$ will be a solution also.   The two components of this *steady-state* solution can be obtained as follows.   First, in order to find i' a solution of

(6.28) $$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{i}{C} = A \varepsilon^{j\omega t}$$

assume
(6.29) $$i' = I' \varepsilon^{j\omega t}$$

where I' is a complex number *not* a function of time.   If Eq. (6.29) is substituted in Eq. (6.28), each term contains $\varepsilon^{j\omega t}$, which may therefore be divided out.   The result is

(6.30) $$I = \frac{A}{R + j\omega L + \dfrac{1}{j\omega C}}$$

Note that the time function is eliminated by this procedure; the result is an algebraic equation relating complex numbers.   The corresponding relation for I'' produced by the component $B \varepsilon^{-j\omega t}$ is obtained by substituting $(-j\omega)$ for $(j\omega)$ and B for A in Eq. (6.30).

(6.31) $$I'' = \frac{B}{R - j\omega L - \dfrac{1}{j\omega C}}$$

Therefore

(6.32) $$i = i' + i'' = \frac{A}{R + j\omega L + \dfrac{1}{j\omega C}} \varepsilon^{j\omega t}$$
$$+ \frac{B}{R - j\omega L - \dfrac{1}{j\omega C}} \varepsilon^{-j\omega t}.$$

This equation, in turn, can be reduced by substituting Eq. (6.27)

* Although i' and i'' are complex numbers, $(i_1 + i_2) = i$ is a real number.

for A and B and by writing

$$(6.33) \quad R + j\omega L + \frac{1}{j\omega C} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \bigg/ \tan^{-1} \frac{\omega L - \dfrac{1}{\omega C}}{R} \equiv Z\underline{/\theta} \equiv Z$$

and

$$(6.34) \qquad\qquad R - j\omega L - \frac{1}{j\omega C} \equiv Z\underline{/-\theta}$$

to

$$(6.35) \qquad\qquad i = \frac{\sqrt{2}\,E}{Z} \cos(\omega t + \phi - \theta)$$

This is the solution obtained in Chap. IV and in Eq. (6.24) above.

Thus from two different points of view it has been shown that the **steady-state** current in a single-mesh circuit can be evolved from the simple relation between the vector (complex-number) expressions of the applied voltage, current, and impedance:

$$(6.22) \qquad\qquad E = ZI$$

The essential point to be noted is that the symbolic expression for the current is equivalent from the analytical point of view to the total instantaneous steady-state current; it therefore satisfies Kirchhoff's first law. Therefore the methods used to obtain general solutions for an n-mesh d-c network (Chap. II) and the methods for using complex numbers to represent *steady-state* a-c voltage and currents can be combined into a process for analyzing an n-mesh network in which there are sinusoidal emfs.* If these emfs are all of the same frequency, the solution is obtained as described below (Sec. 6.05). If emfs of several frequencies are present in the network, the solutions can be assumed to coexist to form the complete current.

6.04. Calculation of Power from Symbolic Voltage and Current.—The analysis of the polar diagram in Chap. V (Sec. 5.04) showed that the average power input P to a two-terminal device is equal to the product of the rms voltage input V, the rms current

---

* Non-sinusoidal periodic emfs can be expressed as the sum of a number of emfs of different frequencies or the methods of the *operational calculus*,[2] closely related to the method described above, may be used.

input I, and the cosine of the phase angle between the voltage and the current. Thus the average power P corresponding to V, I of Fig. 6.03 is

$$P = VI \cos (\theta_v - \theta_i) \qquad \text{watts}$$

It is often useful to calculate a quantity Q volt-amperes such that

$$(6.36) \qquad \sqrt{P^2 + Q^2} = VI \qquad \text{volt-amperes}$$

The quantity Q, called *reactive volt-amperes*, is

$$(6.37) \qquad Q = VI \sin (\theta_v - \theta_i) \qquad \text{volt-amperes}$$
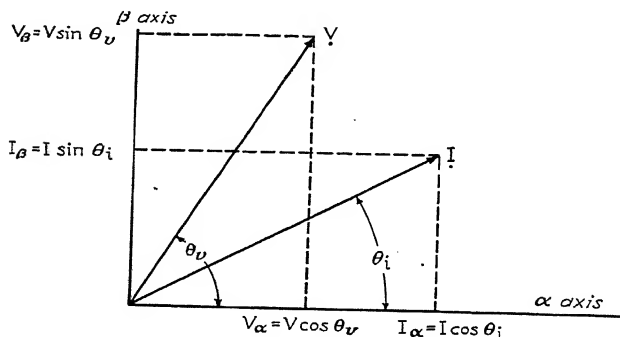
FIG. 6.03.

The voltage V and the current I of Fig. 6.03 can be expressed as complex numbers:

$$(6.38) \qquad \begin{aligned} V &= V_\alpha + jV_\beta = V \cos \theta_v + jV \sin \theta_v \\ I &= I_\alpha + jI_\beta = I \cos \theta_i + jI \sin \theta_i \end{aligned}$$

Calculating P and Q in terms of V_α, V_β, I_α, I_β, the results are

$$P =$$

The product $VI = V_\alpha I_\alpha - V_\beta I_\beta + j(V_\alpha I_\beta + V_\beta I_\alpha)$ contains the same terms as the terms of P and Q of Eq. (6.39). The signs connecting them do not, however, correspond. Terms like those of Eq. (6.39), connected by signs like those of Eq. (6.39) *can*

be obtained by multiplying the voltage V by the *conjugate*
$I = I_\alpha - jI_\beta$ of the current:

$$(6.40) \quad \begin{aligned} VI &= (V_\alpha + jV_\beta)(I_\alpha - jI_\beta) \\ &= (V_\alpha I_\alpha + V_\beta I_\beta) + j(V_\beta I_\alpha - V_\alpha I_\beta) \\ &= P + jQ \end{aligned}$$

This establishes a useful theorem. To find the power P and
the reactive volt-amperes Q corresponding to a symbolic voltage
V and a symbolic current I: (1) multiply the complex number V
by the complex *conjugate* of I; (2) the real term of the product
is the power P; (3) the real part of the imaginary term is the
reactive volt-amperes Q.

6.05. Steady-state Solution for n-mesh Network.—If emfs
e_1, e_2, e_3, . . . , e_n of the same frequency are interconnected
with impedances in an n-mesh network, the system of simul-
taneous linear differential equations that represent the network
comprises n equations of the form

$$(6.41) \quad \begin{aligned} \frac{de_1}{dt} = &\left[L_{11} \frac{d^2 i_1}{dt^2} + R_{11} \frac{di_1}{dt} - \frac{i_1}{C_{11}} \right] \\ &- \left[L_{12} \frac{d^2 i_2}{dt_2^2} + R_{12} \frac{di_2}{dt} - \frac{i_2}{C_{12}} \pm M_{12} \frac{d^2 i_2}{dt^2} \right] - \cdots \\ &- \left[L_{1n} \frac{d^2 i_n}{dt^2} - R_{1n} \frac{di_n}{dt} + \frac{i_n}{C_{1n}} \pm M_{1n} \frac{d^2 i_n}{dt^2} \right] \end{aligned}$$

The methods developed in the last two chapters lead to the
transformation of a set of n such differential equations into a set
of n algebraic equations involving complex numbers:

$$(6.42) \quad \begin{aligned} E_1 &= \quad Z_{11}I_1 - Z_{12}I_2 - Z_{13}I_3 + \cdots - Z_{1n}I_n \\ E_2 &= -Z_{21}I_1 + Z_{22}I_2 - Z_{23}I_3 + \cdots - Z_{2n}I_n \\ &\vdots \\ E_n &= -Z_{n1}I_1 - Z_{n2}I_2 - Z_{n3}I_3 - \cdots + Z_{nn}I_n \end{aligned}$$

Thus the value of the current in the kth mesh is (see Sec. 2.02)

$$(6.43) \quad I_k = \frac{D_k}{D}$$

in which

$$(6.44) \quad D_k = \begin{vmatrix} Z_{11} & \cdots & -Z_{1,k-1} & E_1 & \cdots & -Z_{1n} \\ -Z_{21} & \cdots & -Z_{2,k-2} & E_2 & \cdots & -Z_{2n} \\ -Z_{31} & \cdots & -Z_{3,k-3} & E_3 & \cdots & -Z_{3n} \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ -Z_{n1} & \cdots & -Z_{n,k-1} & E_n & \cdots & +Z_{nn} \end{vmatrix}$$

and

$$(6.45) \quad D = \begin{vmatrix} Z_{11} & - Z_{12} & \cdots & -Z_{1n} \\ -Z_{21} & + Z_{22} & \cdots & -Z_{2n} \\ -Z_{31} & - Z_{32} & \cdots & -Z_{3n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ -Z_{n1} & - Z_{n2} & \cdots & +Z_{nn} \end{vmatrix}$$

Note that, since the real part of the left-hand member of each equation of Eq. (6.42) is equal to the real part of the right-hand member of each equation, and since the imaginary parts chosen in the same manner are equal, Eq. (6.42) represents $2n$ equations. These are required to determine the n magnitudes *and* the n phase angles of the n currents of the network.

The term Z_{kk} is called the *self-impedances* of mesh k. The general form of this term is

$$(6.46) \quad Z_{kk} = R_{kk} + j\omega L_{kk} + \frac{1}{j\omega C_{kk}}$$

in which each term of Z_{kk} represents the sum of the resistances or of the reactances connected in the kth mesh.

The term Z_{hk} is called the *mutual impedance* of meshes h and k. The general form of the term is

$$(6.47) \quad Z_{hk} = R_{hk} + j\omega L_{hk} + \frac{1}{j\omega C_{hk}} \pm j\omega M_{hk}$$

subject to the extensions mentioned in the paragraph above and to the additional comment that M_{hk} is the mutual inductance of mesh h with respect to mesh k. The sign placed before the mutual-inductance term depends upon the relative directions of winding of the two coils. An example of a problem for which

the mesh currents are changed by interchanging the leads of one of two coils is presented in Sec. 6.06.

It is conventional to assume that each mesh current in a network flows in a clockwise direction through the mesh. Thus the term $-Z_{hk}I_k$ is written in the hth mesh equation

$$-\left(R_{hk} + j\omega L_{hk} + \frac{1}{j\omega C_{hk}} \pm j\omega M_{hk}\right)I_k$$

because the term represents a voltage drop in the direction of I_h, and I_k is assumed to flow clockwise in mesh k, which is counterclockwise in mesh h. From the point of view of mesh k the same term is written

$$-\left(R_{kh} + j\omega L_{kh} + \frac{1}{j\omega C_{kh}} \pm j\omega M_{kh}\right)I_h$$

for corresponding reasons. The values of R, L, C, and M are assumed to be constant. They are assumed not to vary for currents flowing through them in opposite directions. It follows that in general

$$(6.48) \qquad\qquad Z_{hk} = Z_{kh}$$

Thus D is symmetrical about the $Z_{11} \ldots Z_{nn}$ diagonal. This fact is useful in the development of the advanced theory of circuits.[3]

Complex-number relations among vector voltages, vector currents, and vector impedances can be derived directly from differential equations that represent electric circuits by substituting I or E for i or e, by substituting $j\omega I$ or $j\omega E$ for di/dt or de/dt, and by substituting $(j\omega)^2 I = -\omega^2 I$ or $(j\omega)^2 E = -\omega^2 E$ for d^2i/dt^2 or d^2e/dt^2. For example, the general network equation (6.41) of this section is

$$(6.41) \qquad \frac{de_1}{dt} = \left(L_{11}\frac{d^2i_1}{dt^2} + R_{11}\frac{di_1}{dt} + \frac{i_1}{C_{11}}\right)$$

$$-\left(L_{12}\frac{d^2i_2}{dt^2} + R_2\frac{di_2}{dt} + \frac{i_2}{C_{12}} \pm M_{12}\frac{d^2i_2}{dt^2}\right)$$

$$\cdots -\left(L_{1n}\frac{d^2i_n}{dt^2} + R_{1n}\frac{di_n}{dt} + \frac{i_n}{C_{1n}} \pm M_{1n}\frac{d^2i_n}{dt^2}\right)$$

When substitutions are made as described above, the result is

$$(6.49) \quad j\omega E_1 = \left(-\omega^2 L_{11} I_1 + j\omega R_{11} I_1 + \frac{I_1}{C_{11}} \right)$$
$$- \left(-\omega^2 L_{12} I_2 + j\omega R_{12} I_2 + \frac{I_2}{C_{12}} \mp \omega^2 M_{12} I_2 \right)$$
$$\cdots - \left(-\omega^2 L I_n + j\omega R_{1n} I_n + \frac{I_n}{C_{1n}} \mp \omega^2 M_{1n} I_n \right)$$

Equation (6.49), divided through by $j\omega$, is

$$(6.50) \quad E_1 = \left(R_{11} + j\omega L_{11} - \frac{j}{\omega C_{11}} \right) I_1$$
$$- \left(R_{12} + j\omega L_{12} - \frac{j}{\omega C_{12}} \pm j\omega M \right) I_2$$
$$\cdots - \left(R_{1n} + j\omega L_{1n} - \frac{j}{\omega C_{12}} \pm j\omega M \right) I_n$$

The coefficients in parentheses in Eq. (6.50) are the coefficients $Z_{11}, Z_{12}, \ldots, Z_{1n}$ defined by Eqs. (6.46) and (6.47). Therefore Eq. (6.50) reduces to

$$(6.51) \quad E_1 = Z_{11} I_1 - Z_{12} I_2 \cdots - Z_{1n} I_n$$

which is the first equation of Eq. (6.42).

In the following section a simple example of the calculation of the steady-state currents in an a-c network is worked out in detail.

Do not forget that complex-number calculations for electric circuits provide solutions only for steady-state currents, in terms of steady-state applied emfs and constant circuit parameters R, L, C, M.

6.06. Example of Calculations Using Symbolic Notation.—The emfs and therefore the currents in this problem have a frequency of 60 cycles per sec. The rms voltage and the phase angle are marked on the wiring diagram, Fig. 6.04. Resistances, inductances, and capacitances are marked in ohms, henrys, and farads, respectively. The equations for the circuit of Fig. 6.04 are, according to Eq. (6.42), noting that I_1 and I_2 are assumed in the clockwise direction

$$E_1 = Z_{11} I_1 - Z_{12} I_2$$
$$E_2 = -Z_{21} I_1 + Z_{22} I_2$$

and the solutions are, according to Eq. (6.43)

$$I_1 = \frac{Z_{22}E_1 + Z_{12}E_2}{Z_{11}Z_{22} - Z_{12}^2} \qquad \text{and} \qquad I_2 = \frac{Z_{11}E_2 + Z_{21}E_1}{Z_{11}Z_{22} - Z_{12}^2}$$

The calculations are listed below; note that $\omega = 2\pi f \approx 377$:

$$Z_{11} = 100 + j\left[(0.2)(377) - \frac{10^6}{(30)(377)}\right] = 100.8\underline{/-7.4°}$$

$$Z_{22} = (50 + 50) + j\left[(0.2)(377) + (0.5)(377) - \frac{10^6}{(30)(377)}\right] = 202\underline{/60.3°}$$

$$Z_{12} = j\left[+(0.1)(377) - \frac{10^6}{(30)(377)}\right] = 50.7\underline{/-90°}$$

$$Z_{22}E_1 + Z_{12}E_2 = 15,070 + j17,550 = 23,132\underline{/49.35°}$$

$$Z_{11}Z_{22} - Z_{12}^2 = 14,848 + j16,240 = 22,005\underline{/47.5°}$$

$$I_1 = \frac{23,132\underline{/49.35°}}{22,005\underline{/47.5°}} = 1.05\underline{/1.8°}$$

$$Z_{11}E_2 + Z_{21}E_1 = 1300 + j4930 = 5,098\underline{/75.3°}$$

$$I_2 = \frac{5,098\underline{/75.3°}}{22,005\underline{/47.5°}} = 0.23\underline{/27.8°}$$
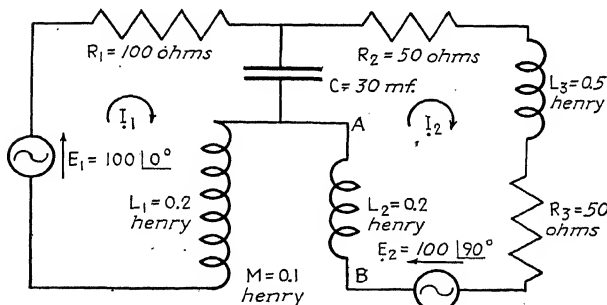
FIG. 6.04.

Thus the steady-state rms current in mesh 1 is 1.05 amp.; this current leads the applied emf E_1 by an angle of 1.8 deg. The current in mesh 2 is 0.23 amp. rms; it lags the voltage E_2 by an angle of 62.2 deg.

It was assumed in the solution presented above that $j\omega M = j37.7$ ohms. If the circuit is set up in the laboratory so that the emf induced in coil AB (Fig. 6.04) is in the sense represented by taking M positive, the measured currents are those calculated above. If, then, the connections to A and B are interchanged, a different pair of mesh currents is found to exist in meshes 1 and 2; these currents correspond to values which can be calculated by assuming $j\omega M = -j37.7$ ohms. The essential point is that interchanging the connections to A and B causes the induced emf in this coil to be applied in the reverse sense to mesh 2; and this reacts on mesh 1 to cause a change

in the value of I_1 as well as a change in I_2.   The currents are

$$I_1 = 0.88\underline{/7.8°} \qquad \text{and} \qquad I_2 = 0.09\underline{/-93.5°}$$

The student may check these values as a practice problem.

References

1. See, for example, *Advanced Calculus;* E. B. Wilson; Chap. VI, Ginn and Company, Boston, 1912.
2. *Electric Circuit Theory and the Operational Calculus;* John R. Carson; McGraw-Hill Book Company, Inc., New York, 1927.
3. *Ibid.*, p. 8.

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Problems

6.01. Given the following relations:

$$M = \frac{C}{D} = \frac{C\underline{/\theta_C}}{D\underline{/\theta_D}} \qquad A = \tfrac{1}{2}(1 + M)\varepsilon^\gamma = A\underline{/\phi_A}$$

$$\gamma = \alpha + j\beta \qquad\qquad B = \tfrac{1}{2}(1 - M)\varepsilon^{-\gamma} = B\underline{/\phi_B}$$

$$\mu = \theta_C - \theta_D \qquad\qquad \phi = \phi_A - \phi_B$$

show that

$$A = \tfrac{1}{2}\varepsilon^\alpha \sqrt{1 + M^2 + 2M \cos \mu}$$

$$B = \tfrac{1}{2}\varepsilon^{-\alpha} \sqrt{1 + M^2 - 2M \cos \mu}$$

$$\phi_A = \beta + \tan^{-1}\left(\frac{M \sin \mu}{1 + M \cos \mu}\right)$$

$$-\phi_B = \beta + \tan^{-1}\left(\frac{M \sin \mu}{1 - M \cos \mu}\right)$$

$$\phi = 2\beta + \tan^{-1}\left(\frac{2M \sin \mu}{1 - M^2}\right)$$

6.02. If

$$\sqrt{\frac{R + j\omega L}{G + j\omega C}} = R' + jX'$$

calculate R' and X' as functions of R, L, G, C, and ω.   Specify one or more relations among R, L, G, C such that $X' = 0$.

6.03. Show that the conjugate of any algebraic expression involving complex numbers A_1, $A_2 \ldots$, A_n combined by addition, subtraction, multiplication, and division can be obtained by substituting for the A's, their conjugates.

6.04. Calculate the average power input to a device when the current through the device and the voltage between its terminals are

$$i = I \sin (\omega t + \phi)$$
$$v = V \sin \omega t$$

Calculate the impedance, effective resistance, and effective reactance of the device. Represent the current and voltage by complex numbers and calculate average power input, impedance, effective resistance, and effective reactance.

6.05. Calculate the current that would flow in the circuit shown below.
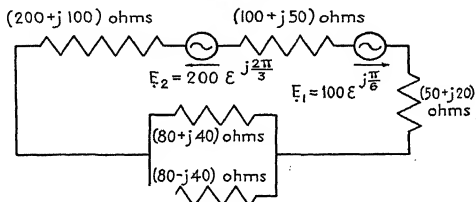
PROB. 6.05.

6.06. Three impedances A, B, C have resistances of 5, 8, 3 ohms and reactances of 14, 0, -10 ohms. Calculate (*a*) the impedance of A, B, and C connected in series; (*b*) the impedance of A, B, and C connected in parallel; (*c*) the admittance of A, B, and C connected in parallel.

6.07. A coil of resistance R and inductance L is connected in parallel with a condenser of capacitance C and leakage G. What is the impedance of the combination? What is its admittance? At what frequency is the effective reactance of the combination equal to zero? At what frequency is its impedance maximum?

6.08. What load impedance $Z_L = R_L + jX_L$ should be connected to a generator having an internal impedance $Z_G = R_G + jX_G$ in order to deliver as much power to Z_L as possible?

6.09. If the *magnitude* of an impedance Z_L can be varied but its phase angle cannot, to what value should Z_L be adjusted in order that maximum power shall be taken from a generator having an internal impedance Z_G?

6.10. Under what conditions will the current in Z_R of Fig. 6.10 be zero?
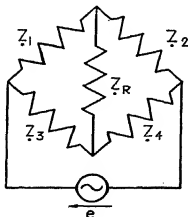
PROB. 6.10.

6.11. Show that, if the response of an indicator Z_R in the figure for Problem 6.10 is proportional to the power delivered to it, the sensitivity of the bridge is maximum if: (1) the four bridge impedances are equal in magnitude, and (2) the impedance Z_R is equal to the impedance of a bridge arm.

6.12. Design a bridge for measuring inductances from 100 millihenrys to 10 henrys, using a source of 1,000 cycles a.c. and a pair of headphones as null indicator.

6.13. Calculate Z_a, Z_b, Z_c in terms of Z_1, Z_2, Z_3 so that the two networks shown below become equivalent to each other in the sense indicated by the terminal markings A, B, C, D. If $Z_1 = Z_2$, what are the values of Z_a, Z_b, Z_c for equivalence?
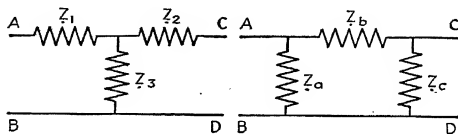
PROB. 6.13.

CHAPTER VII

POLYPHASE CIRCUITS
AND SYMMETRICAL COMPONENTS

7.01. Polyphase Alternating Currents.—A coil of wire and one or more pairs of magnetic poles can be so arranged that relative rotating motion between the coil and the poles causes the induction of a sinusoidal emf in the coil as noted in Secs. 3.02 and 5.02 and discussed in detail in Chap. XII. In most alternators the *armature coil,* in which the alternating emf is induced, is stationary while the magnetic *field poles*, produced by the flow of direct current in coils of wire, are caused to rotate by some mechanical device such as a steam turbine, water turbine, or internal combustion engine. Both the armature and the field windings are mounted on structures of laminated iron or steel and the annular air gap between the outer surface of the field poles and the inner surface of the armature structure is made as small as practicable because these procedures improve the operating characteristics of the alternator.[1] If two or more coils are placed on the cylindrical armature structure, an alternating emf is induced in each coil. The coils can be so arranged that the emfs induced in them are sinusoidal. Electric energy can be supplied by each coil to a circuit connected to it. Under certain conditions it is more efficient and convenient to interconnect terminals of the coils of the generator and to conduct current from the generator to the load by means of a number of conductors less than twice the number of armature coils. It is the purpose of this chapter to specify the most common of these conditions and to develop means for analyzing the circuits, including the interconnected coils of the generator, the leads from the generator to the load, and the load.

An a-c generator having two armature coils so designed that the emfs induced in them are equal in magnitude and differing in phase by 90 degrees is called a *two-phase alternator*. A *three-phase alternator* comprises three coils so arranged that the three

emfs induced in them are equal in magnitude; their relative phase angles can be represented by the angles 0, 120, 240 degrees. Each is an example of *polyphase alternators*. The system comprising the generator, transmission line, and loads is called a polyphase a-c system. *Nearly all a-c generating and distributing systems in this country are three-phase systems.* Thus the analysis of polyphase alternating currents is an important electrical engineering problem, to which this chapter is a short introduction.

Polyphase systems are used in preference to single-phase systems for the generation, distribution, and use of electrical energy because: (1) a polyphase generator is less costly than a single-phase generator of the same output, voltage, and speed; (2) a polyphase generator has higher efficiency and better voltage regulation than a single-phase generator of the same general design and rating; (3) a polyphase transmission line is less costly than the corresponding single-phase line that will transmit the same power at the same efficiency; (4) polyphase motors[2] are less costly and in general have better operating characteristics than single-phase motors. Finally, single-phase alternating current can be obtained from a polyphase system when it is convenient to use, chiefly for small loads such as household appliances.

7.02. Symbols and Nomenclature of Polyphase Quantities.— Since the coils of a polyphase alternator and the branches of a polyphase load may be interconnected, the emfs, voltage drops, and currents in a polyphase system must be designated unambiguously to avoid errors in calculation. If the magnitudes of these quantities and their phase angles with respect to a chosen reference are known, a polar diagram can be constructed and interpreted. However, if the currents are unknown, and algebraic equations involving the complex quantities are solved to obtain their values, care must be taken to use symbols that indicate clearly just what they represent in the analytical process. The conventions for the symbolic representation of polyphase quantities that are used in the literature vary, unfortunately, from one author to another. The result is confusing chiefly because each worker acquires the habit of using a particular method of designation and then finds it difficult to follow the work of someone else who has chosen to learn and to use a differ-

ent set of symbols.   The list of conventions given below is there-
fore just as arbitrary as that presented in any other textbook.*

In d-c circuit theory, the terminology is not likely to be
ambiguous because of the unidirectional property of direct cur-
rent.   Having assumed that the current flow in a d-c source is
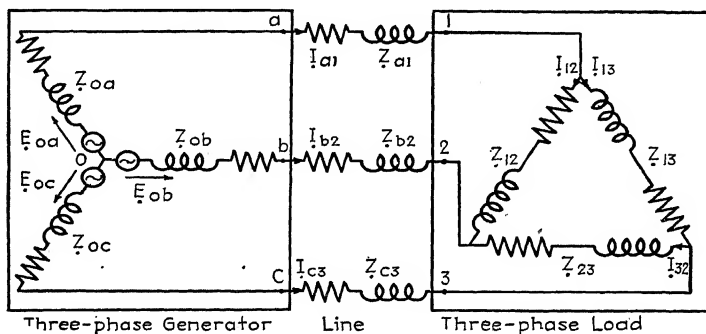
Three-phase Generator   Line   Three-phase Load
Fig. 7.01.

from the negative terminal to the positive terminal of the source
and having defined a d-c emf as the rise in potential from the
negative to the positive terminal of a source through which no
current flows, the notation for the analytical solution of d-c
problems is straightforward and unambiguous.

In a-c circuit theory the problem is more difficult.   The direc-
tion of a-c flow in a wire is in general parallel to the axis of the
wire, but the *sense* of the flow is indeterminate because the
sense varies cyclically with time.   Similarly, the sense in which
the electric potential rises between two terminals *A*, *B* of a
device connected in an a-c circuit is from *A* to *B* at one instant
and from *B* to *A* at an instant 1 half cycle later, and therefore
indeterminate.   These difficulties may be overcome by choosing
symbols related to the chosen reference quantity, as described
below.

Assume that the circuit under consideration is that shown in
Fig. 7.01.   This is a particular example of a polyphase circuit
consisting of a three-phase generator connected through a three-

---

* The authors have never known anyone who has consistently used the
same notation for all of his notes, lectures, and technical papers, present
company *not* excepted.

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wire transmission line to a three-phase load. Each branch *Oa, Ob, Oc,* represents the emf and impedance of a particular coil of the generator. The emfs of the three coils of a three-phase generator are usually of equal magnitude and their phase angles differ by ±120 degrees. One, say that in the branch *Ob,* can be chosen as reference. Then its instantaneous value is

$$(7.01) \qquad e_{ob} = \sqrt{2}\, E \cos \omega t$$

of which the graphical representation is shown in Fig. 7.02. The complex number $E_{ob} = E/\underline{0}$ degrees corresponding to Eq. (7.01)
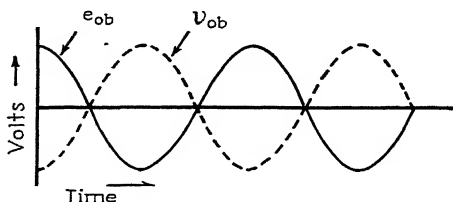
FIG. 7.02.

is taken as the emf from O to b and the sense of this emf is taken as the sense from O to b as indicated by the arrow near the symbol E_{ob} in Fig. 7.01. In general the procedure is as follows: (1) express the instantaneous emf in the form

$$e_{hk} = \sqrt{2}\, E \cos (\omega t - \phi),$$

where the subscript hk indicates that terminal h of the emf remains *negative* with respect to terminal k for one quarter cycle before and after the instant for which $\omega t - \phi = 0$; (2) use the symbol E_{hk} as the complex-number representation of the sinusoidal quantity e_{hk}; (3) the sense of the emf is specified by the subscript of E_{hk} or it may be further indicated by an arrow in the circuit pointing in the sense $h \rightarrow k$.

Throughout this chapter the symbols e_{hk} and E_{hk} are used to represent an *electromotive force* in the manner described above. The symbols v_{hk} and V_{hk} are used to represent terminal voltages or impedance drops. These latter quantities are taken as positive when they represent a quantity of the form

$$(7.02) \qquad = \sqrt{2}\, V \cos (\omega t - \phi - 180°) = -\sqrt{2}\, V \cos (\omega t - \phi)$$

of which an example v_{ob} is represented graphically by the dotted curve of Fig. 7.02 *if there is no current in the branch Ob.* The complex number V_{ob} is in this case equal to $-E_{ob} = E/180°$. In general the procedure is as follows: (1) express the instantaneous voltage in the form $v_{hk} = \sqrt{2} \, V \cos (\omega t - \phi - 180°)$, where the subscript *hk* indicates that terminal *h* remains *positive* with respect to terminal *k* for one quarter cycle before and after the instant for which $\omega t - \phi = 0$; (2) use the symbol V_{hk} as the complex-number symbol of the sinusoidal quantity v_{hk}; (3) the sense of the voltage is specified by the subscript of V_{hk} or it may be further indicated by an arrow on the circuit diagram pointing in the sense $h \to k$.

The notion of a voltage drop or voltage rise in a particular sense can be associated with constant complex numbers V_{hk} and E_{hk} only if these symbols are derived from an analysis of the time functions [such as Eqs. (7.01) and (7.02) above] *that they represent.*

The initial choice of a sense of the flow of current in a conductor or mesh is arbitrary. This sense can be indicated by an arrow in the diagram or by a double subscript such as I_{a1} referring to the assumed sense $a \to 1$, Fig. 7.01.

The voltage across an impedance has a positive sense in the direction of the current, in accordance with the conventions described above. Thus it is permissible to assume that the line currents of Fig. 7.01 I_{a1}, I_{b2}, I_{c3} all flow away from *O*, because the Kirchhoff first law

$$(7.03) \qquad\qquad I_{a1} + I_{b2} + I_{c3} = 0$$

shows that some parts, both real and imaginary, are negative; the circuit analysis leading to solutions for I_{a1}, I_{b2}, I_{c3} will show just what the real and imaginary parts of the currents are and how they combine to satisfy Eq. (7.03).

Although Fig. 7.01 represents a particular kind of polyphase circuit, the nomenclature relating to all polyphase circuits can be illustrated in terms of the quantities shown in Fig. 7.01.

1. I_{a1}, I_{1a}, I_{b2}, I_{2b}, I_{c3}, I_{3c}, are examples of *generator phase currents.*

2. I_{12}, I_{13}, I_{23}, and the variations like those given in 1 are called *phase currents in the load.*

3. E_{ob}, E_{oa}, E_{oc}, . . . , are *phase emfs* in the generator.

4. I_{a1}, I_{b2}, I_{c3}, . . . , are called *line currents*.

5. V_{ac}, V_{bc}, V_{ac}, . . . , are called *line voltages*.

6. $V_{bo} = E_{ob} - Z_{ob}I_{b2}$, . . . , are called *phase voltages*.

7. The difference between the phase angles of V_{bo} and I_{b2} is called the *phase angle* of phase *Ob*.

Many relations among the quantities so defined can be seen by examining the figure. A few examples follow. The generator phase currents *are* equal respectively to the line currents. The generator phase voltages *are not* equal to the line voltage. For example,

$$(7.04) \qquad V_{ab} = - \qquad\qquad Z_{ob}I_{b}$$

The load phase currents *are not* equal to the line currents. The load phase voltages *are* equal to the line voltages. The total power generated is the sum of (1) the product of the real parts and (2) the negative of the product of the imaginary parts of E and I for all of the products $E_{ob}I_{b2}$, $E_{oa}I_{a1}$, and $E_{oc}I_{c3}$. This total power input is dissipated in the resistances of the generator phases $(R_{oa}I_{a1}^2 + R_{ob}I_{a2}^2 + R_{oc}I_{c3}^2)$, in the line $(R_{a1}I_{a1}^2 + R_{b2}I_{a2}^2 + R_{c3}I_{c3}^2)$ and in the load $(R_{13}I_{13}^2 + R_{32}I_{32}^2 + R_{12}I_{12}^2)$. If there were emfs in the load, power would be converted in their sources to forms other then heat.

It is clear that, unless the phase voltages, line voltages, line currents, etc., are in each case equal and symmetrical with respect to phase, the solution of polyphase circuits is at best very tedious. General polyphase problems, of which the circuit in Fig. 7.01 is an example, are not discussed further. The next two sections are devoted to commonly used circuits in which the equality of certain quantities and their symmetry from the point of view of the polar diagram representing them lead to relatively simple solutions. Two-phase systems are described briefly in Sec. 7.03; various aspects of the nearly universally used three-phase systems are discussed in Secs. 7.04 and 7.05.

7.03. Two-phase Systems.—The emfs generated in the two windings or *phases* of a two-phase alternator are equal in magnitude and they differ in phase by 90 degrees. The four leads may be brought out separately as shown in Fig. 7.03 *left*, or two may be connected together, thus reducing the number of terminals of the generator to three as shown in Fig. 3, *right*. The emfs generated in the two phases are, taking e_{21} as reference,

(7.05) $= \sqrt{2}\, E \cos \omega t$

(7.06) $= \sqrt{2}\, E \cos (\omega t - 90°)$

as shown in Fig. 7.04.   These emfs can be represented by complex
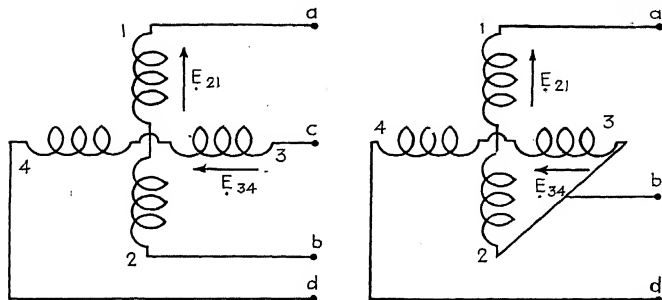
FIG. 7.03.
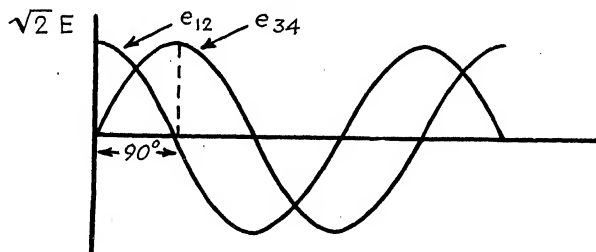
FIG. 7.04.

numbers as follows:

$$E_{21} = E\underline{/0°}$$
$$E_{34} = E\underline{/-90°}$$

Thus the line voltages for the four-wire system are

neglecting the impedance drops within the generator.   In Fig. 7.03, *right,* the line voltages are

$$V_{ab} = E_{21} = E\underline{/0°}$$
$$V_{db} = E_{34} = E\underline{/-90°}$$
$$V_{ad} = E_{21} - E_{34} = E\underline{/0°} + E\underline{/90°} = \sqrt{2}\,.$$

neglecting the impedance drops in the phase windings. If V_{12} and V_{43} are the phase voltages including the impedance drops, if I_{1a} and I_{4d} are the phase currents, and if θ_1 and θ_4 are the respective phase angles of these currents with respect to these voltages, the power output of the generator is

$$(7.07) \qquad P = V_{12}I_{1a} \cos \theta_1 + V_{43}I_{4d} \cos \theta_4 \qquad \text{watts}$$

for either the three-wire or the four-wire connections. When the phase voltages and the phase currents have, respectively, equal magnitudes and phase differences of 90 degrees, the adjective *balanced* is used to describe the currents and voltages and the conditions of operation of the generator. Thus when the conditions of operation of the generator are balanced

$$(7.08) \qquad P = 2VI \cos \theta \qquad \text{watts}$$

V is the rms phase voltage, I the rms phase current, and θ is the phase angle between phase voltage and phase current. In the three-wire two-phase system (Fig. 7.03, *right*), the line current flowing into terminal b is equal to the resultant of the currents flowing out of a and d, and the latter are the phase currents. Therefore, if the phase currents in a three-wire system are balanced

$$(7.09) \qquad \qquad \qquad \qquad \overline{} = \sqrt{2}\, I \qquad \text{amperes}$$

where I is the magnitude of the phase current. Similarly the voltage across the wires a and d in a balanced system is

$$(7.10) \qquad V_{da} = \qquad \qquad \overline{7_{21}^2} = \sqrt{2}\, V \qquad \text{volts}$$

where V is the magnitude of the phase terminal voltage.

It is clear that the two-phase four-wire system is equivalent to two single-phase systems. The three-wire system involves properties that are different from those of single-phase systems. For example, if the terminal-phase voltages and phase currents of a three-wire two-phase generator are balanced, the terminal-phase voltages and phase currents of a two-phase load connected to the generator by means of a three-wire transmission line having wires of equal and not negligible impedance are *not* balanced. This follows from the fact that the line currents are not equal, so that the voltage drops in the three wires are different. The

student should prove these statements by drawing a polar diagram representing the pertinent alternating quantities related to the generator, the transmission line, and the load.

7.04. Three-phase Systems; Methods of Connection.—The emfs of the three phases of a three-phase alternator are equal in magnitude and they differ in phase by ±120 degrees. Three-phase generators, transmission lines, and motors have properties
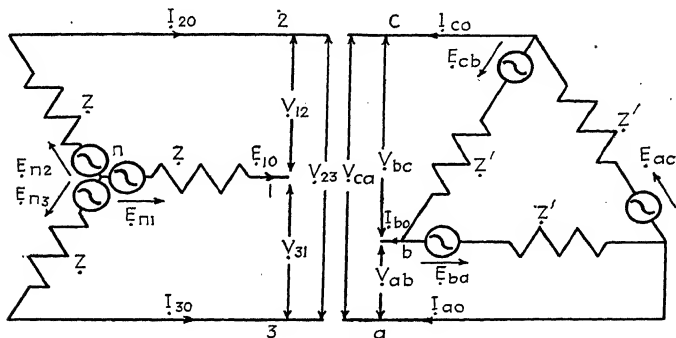
FIG. 7.05.

that lead to high efficiency, high stability, good regulation, great flexibility, and relatively low cost. Therefore three-phase systems are far more commonly used than single-phase or two-phase systems for generating, distributing, and using electric power.

Three-phase generators or loads are usually connected in one of the two ways shown in Fig. 7.05.* The (Y) or *star* connection is shown on the left; the *delta* (Δ) or *mesh* connection on the right. These connections were discussed briefly but not named in Sec. 7.02 (see Fig. 7.01). If it is assumed that

$$
\begin{array}{ll}
& I_{10} = I\underline{/\theta} \\
\text{and} & I_{20} = I\underline{/\theta + 120°} \\
& I_{30} = I\underline{/\theta + 240°}
\end{array}
$$

(7.11)

$$E_{n3} =$$

---

* Note that the symbol —⋀⋀⋀⋀⋀— marked Z is used here and subsequently as a general symbol for impedance of the form $Z = R + jX$, in conformity with modern textbooks, handbooks, and technical periodicals.

the terminal voltages of the Y are balanced.   Similarly, if

$$(7.12) \qquad \begin{aligned} E_{ac} &= E'/\underline{120°} \\ E_{cb} &= E'/\underline{240°} \end{aligned} \qquad \text{and} \qquad \begin{aligned} I_{co} &= I'/\underline{\theta' + 120°} \\ I_{ao} &= I'/\underline{\theta' + 240°} \end{aligned}$$

the terminal voltages of the $\triangle$ are balanced.   For these balanced conditions note that in the $\triangle$ each line current is the resultant of two phase currents

$$(7.13) \qquad \begin{aligned} I_{co} &= I_{bc} - I_{ca} \\ I_{bo} &= -I_{bc} + I_{ab} \\ I_{ao} &= I_{ca} - I_{ab} \end{aligned}$$

while in the Y each line voltage is the resultant of two phase voltages.

$$(7.14) \qquad \begin{aligned} V_{12} &= -V_{n1} + V_{n2} = -(+ZI_{10} - E_{n1}) + (ZI_{20} - E_{n2}) \\ V_{31} &= -V_{n3} + V_{n1} = -(+ZI_{30} - E_{n3}) + (ZI_{10} - E_{n1}) \\ V_{23} &= -V_{n2} + V_{n3} = -(+ZI_{20} - E_{n2}) + (ZI_{30} - E_{n3}) \end{aligned}$$

On the other hand, in the Y the line currents are equal to the phase currents, while in the $\triangle$ the line voltages are equal to the phase voltages.   Note that the resultant obtained by combining the three complex quantities of any of the sets in Eqs. (7.11) to (7.14), is zero.   These are examples of the general result that the sum of three quantities representable on a polar diagram by three radial lines of equal length whose angles with respect to each other are ± 120 degrees is identically zero.

An important conclusion based upon this identity is that, when three balanced voltages V_1, V_2, V_3 are connected in $\triangle$, the resultant $(V_1 + V_2 + V_3)$ voltage acting about the loop is zero, *if the phase sequences are identical, or equivalent to those listed in Eq. (7.12) above.*   Thus the resultant of $E/\underline{0°}$, $-E/\underline{120°}$, $E/\underline{240°}$— obtained by reversing the sign of one quantity , *i.e.*, by reversing the connections to the terminals of one phase, *is not* zero.

In the balanced Y, the line voltages are those listed in Eq. (7.14) above.   Note that in each case, a line voltage (such as V_{12}) is equal to the resultant obtained by combining a phase

voltage (V_{n2}) and the negative of another phase voltage $(-V_{n1})$. The phase voltages are equal in magnitude, and they differ in phase by 120 degrees as shown in Fig. 7.06. *The line voltage in a balanced Y has therefore a magnitude of* $\sqrt{3}$ *times the magnitude of the phase voltage.*

In the balanced Δ, the line currents are those listed in Eq. (7.13) above. Note that in each case, a line current (such as I_{co}) is equal to the resultant obtained by combining a phase current (I_{bc}) and the negative of another phase current $(-I_{ca})$. The phase currents in turn are of the form

$$(7.16) \qquad I_{bc} = \frac{E'}{Z'} \qquad V'$$

Fig. 7.06.

Since the E's and V's are assumed to be balanced, the phase currents are balanced also. Thus each line current from a balanced Δ is the resultant of two equal complex quantities (phase currents) equal in magnitude and different in phase by 60 degrees. *Therefore the magnitude of a line current in a balanced* Δ *is* $\sqrt{3}$ *times the magnitude of a phase current.*

In general the average power input or output of a three-phase Y or Δ, balanced or unbalanced, is

$$(7.17) \qquad P = V_1 I_1 \cos\theta_1 + V_2 I_2 \cos\theta_2 + V_3 I_3 \cos\theta_3 \qquad \text{watts}$$

where the V's and I's are the rms *phase* voltages and currents of phases 1, 2, 3, and each θ is the difference in phase between the V and I with which it is written. When the circuit, Δ or Y, is balanced the phase voltages, currents, and phase angles are equal so that

$$(7.18) \qquad P = 3 V_p I_p \cos\theta_p \qquad \text{watts}$$

where the subscripts p are used to emphasize the fact that the quantities so marked are *phase* quantities. If the circuit is a balanced Y (using the subscript l for "line")
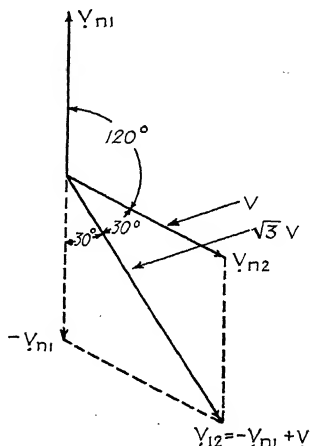
$$(7.19) \qquad V_p = \frac{V_l}{\sqrt{3}}; \qquad I_p = I_l$$

so that

$$(7.20) \qquad P = \sqrt{3}\, V_l I_l \cos \theta_p \qquad \text{watts}$$

If the circuit is a balanced Δ,

$$(7.21) \qquad V_p = V_l; \qquad I_p = \frac{I_l}{\sqrt{3}}$$

so that

$$(7.22) \qquad P = \sqrt{3}\, V_l I_l \cos \theta_p \qquad \text{watts}$$

the power is calculated in the same manner for either Δ or Y.

A balanced Δ is said to be *equivalent* to a balanced Y if the two circuits, considered simply as two three-terminal a-c networks, are interchangeable from the electrical point of view. It follows from the discussion presented above that the phase emfs and currents for equivalent balanced Δ and Y are

	Δ	Y
Phase emf.........................	$\sqrt{3}\,E$	E
Phase currents....................	I	$\sqrt{3}\,I$

if the no-load terminal voltages are $\sqrt{3}\,E$, and the load currents are $\sqrt{3}\,I$. The phase resistances must be such that three times the product of the square of the rms phase current and the phase resistance is the same for Δ and Y

$$(7.23) \qquad \begin{array}{ll} \text{Phase resistance for Y} \ . \ . \ . \ R & \text{ohms} \\ \text{Phase resistance for } \Delta \ . \ . \ . \ 3R & \text{ohms} \end{array}$$

Since, according to Eq. (7.19) the difference in phase between the phase voltage and the phase current must be the same for Δ and Y for all loads, the ratio of the phase reactance to the phase resistance must be the same for Δ and Y. Therefore

$$(7.24) \qquad \begin{array}{ll} \text{Phase reactance for Y} \ . \ . \ . \ X & \text{ohms} \\ \text{Phase reactance for } \Delta \ . \ . \ . \ 3X & \text{ohms} \end{array}$$

An a-c wattmeter indicates the average value of the product of the instantaneous voltage connected to its *potential* terminals

and the instantaneous current flowing through its *current coil*. The total power input or output of a three-phase three-wire system can be measured by connecting two wattmeters as shown in Fig. 7.07. The student should prove this statement and show how the two wattmeter readings are to be combined (added or subtracted) to obtain the average three-phase power.

Note that the material presented in this chapter is based upon the assumption that the steady-state currents and voltages are
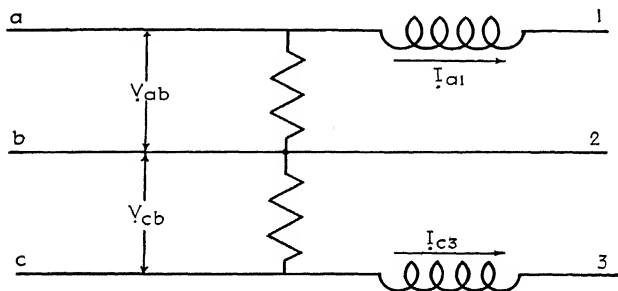
Fig. 7.07.

sinusoidal. In most power circuits the voltages and currents are sufficiently nearly sinusoidal (usually 60 cycles or 25 cycles) that the analyses given here lead to accurate results. However, for precise results, it may be necessary to consider the small emfs and currents of frequencies that are integral multiples of the nominal or *fundamental* frequencies. Such multiple frequencies are called *harmonic* frequencies. Other problems, such as the calculation of the immediate effects of connecting or disconnecting a part of an a-c network, require an analysis of the transient currents which precede the steady state.

If three-phase circuits are unbalanced, the analysis usually requires a general solution of the mesh equations of the form described in Sec. 6.05. This procedure is often tedious. One method that has been developed to circumvent the difficulty is described in the next section.

7.05. Three-phase Systems; Symmetrical Components.— When a generator and load, both arranged in Y, are interconnected by means of three transmission wires the system will be balanced if (1) the generator emfs are balanced, (2) the emfs in

the receiver, if any, are balanced, (3) the three phases of the generator have equal resistances and reactances, (4) the three phases of the load have equal resistances and reactances, and (5) the three wires of the transmission line have equal resistances and reactances. *Under these conditions, the difference of potential between the neutral points (corresponding to n, Fig. 7.05) is zero at all instants of time.* The three-phase circuit can therefore be solved by analyzing the single-phase circuit consisting of one phase of the generator, one transmission wire, and one phase of the load, as shown in Fig. 7.08. Furthermore a balanced Δ generator or load can be replaced by an equivalent Y as shown
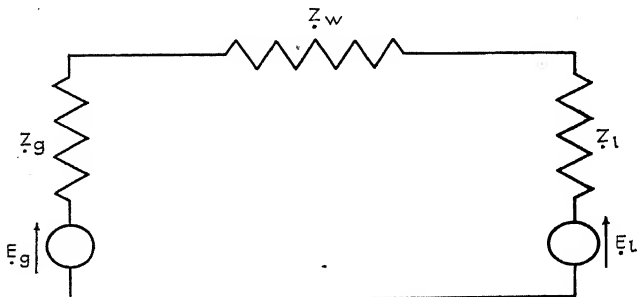
Fig. 7.08.

in Sec. 7.04 so that the scheme here described can be used for any balanced three-phase circuit. It is clear that the single-mesh circuit shown in Fig. 7.08 is much easier to solve than the corresponding complete three-phase circuit involving two meshes and six emfs.

A polyphase circuit may be unbalanced in one or more of several ways. For example, the generator emfs may be unbalanced, the line impedances may be unbalanced, the load may be unbalanced, or one or more interconnected circuits of a polyphase system may be unbalanced. Circuits in which there are one or two points of unbalance are now commonly solved by the method of *symmetrical components*.[3]

This section comprises a short introduction to symmetrical components in three-phase systems. The chief importance of the use of this method is its applicability to the solution of complex networks.[4] The discussion here is elementary. It is

assumed in the following that only steady-state, fundamental $\left(\text{frequency } \dfrac{\omega}{2\pi}\right)$ currents are present in the circuit, and that the impedances are constant. The applications of this method will be studied in subsequent courses on the theory of a-c networks.

Three emfs

$$(7.25) \qquad \begin{aligned} e_a &= \sqrt{2}\, E_a \cos \omega t \\ e_b &= \sqrt{2}\, E_b \cos (\omega t - \theta) \\ e_c &= \sqrt{2}\, E_c \cos (\omega t - \theta - \phi) \end{aligned}$$

are conventionally described as having the *phase sequence a, b, c* because each traverses its first positive maximum in the chronological order $t_a = 0,\ t_b = \dfrac{\theta}{\omega},\ t_c = \dfrac{\theta + \phi}{\omega}.$ Note that these three emfs are not balanced three-phase emfs unless

$$(7.26) \quad E_a = E_b = E_c; \qquad \theta = 120°; \qquad \theta + \phi = 240°$$

The nine symmetrical components corresponding to the three emfs described by Eq. (7.25) are derived below.

Next consider the complex number a defined as

$$(7.27) \qquad a = 1\underline{/120°} = \varepsilon^{j\frac{2\pi}{3}} = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

It is clear that a is one of the cube roots of one and that successive powers of a form a recurrent series

$$(7.28) \quad a^0 = 1 + j0; \qquad a = -\frac{1}{2} + j\frac{\sqrt{3}}{2};$$

$$a^2 = -\frac{1}{2} - j\frac{\sqrt{3}}{2}; \qquad a^3 = 1 + j0$$

Thus a *balanced* system of three-phase emfs

$$(7.29) \qquad \begin{aligned} e_a &= \sqrt{2}\, E \sin \omega t \\ e_b &= \sqrt{2}\, E \sin (\omega t - 120°) \\ e_c &= \sqrt{2}\, E \sin (\omega t - 240°) \end{aligned}$$

having the phase sequence *a, b, c*, may be represented by

$$(7.30) \qquad \begin{aligned} E_a &= a^0 E \equiv E_{a1} \\ E_b &= a^2 E \equiv E_{b1} \\ E_c &= a E \equiv E_{c1} \end{aligned}$$

Such a system is called a *positive-sequence system of symmetrical components.* The notation E_{a1}, E_{b1}, E_{c1} is now commonly used for the positive-sequence system of symmetrical components. In this case the emfs of Eq. (7.29) can be represented by the positive-sequence system of Eq. (7.30) *alone.* Similarly a *balanced* system

$$
(7.31) \qquad
\begin{aligned}
e_a' &= \sqrt{2}\, E' \sin \omega t \\
e_b' &= \sqrt{2}\, E' \sin (\omega t + 120°) \\
e_c' &= \sqrt{2}\, E' \sin (\omega t + 240°)
\end{aligned}
$$

can be represented by the *negative-sequence system of symmetrical components*

$$
(7.32) \qquad
\begin{aligned}
E_a' &= a^0 E' \equiv E_{a2} \\
E_b' &= a E' \equiv E_{b2}
\end{aligned}
$$

the notation E_{a2}, E_{b2}, E_{c2} being the notation now commonly used. In both Eq. (7.30) and Eq. (7.32) note that the resultant of the three emfs is zero because, from Eq. (7.28)

$$(7.33) \qquad a^0 + a^1 + a^2 = 0 \qquad \text{or} \qquad 1 + a + a^2 = 0$$

Finally a "three-phase" emf of the form

$$
(7.34) \qquad
\begin{aligned}
e_b'' &= \sqrt{2}\, E'' \\
e_c'' &= \qquad\qquad\quad \omega t
\end{aligned}
$$

can be represented by the *zero-sequence system of symmetrical components*

$$
(7.35) \qquad
\begin{aligned}
E_a'' &= E'' \equiv E_{a0} \\
E_b'' &= E'' \equiv E_{b0} \\
E_c'' &= E'' \equiv E_{c0}
\end{aligned}
$$

The examples of the symmetrical components given above can be generalized.[3] Any set of complex numbers E_a, E_b, E_c can be represented by a combination of nine symmetrical components; three marked with the subscript 0 to denote the zero-sequence components; three marked 1 for the positive-sequence components; and three marked 2 for negative-sequence components

$$
(7.36) \qquad
\begin{aligned}
E_a &= E_{a0} + E_{a1} + E_{a2} \\
E_b &= E_{b0} + E_{b1} + E_{b2} \\
E_c &= E_{c0} + E_{c1} + E_{c2}
\end{aligned}
$$

Values of the nine components, generalized from the results shown in Eqs. (7.30), (7.32), (7.35), can be substituted in Eq. (7.36)

$$
\begin{aligned}
E_a &= E_{a0} + a^0 E_{a1} + a^0 E_{a2} \\
E_b &= E_{a0} + a^2 E_{a1} + a E_{a2} \\
E_c &= E_{a0} + a E_{a1} + a^2 E_{a2}
\end{aligned}
$$

(7.37)

The symmetrical components E_{a0}, E_{a1}, E_{a2} can be expressed in terms of the three given complex numbers E_a, E_b, E_c by noting Eq. (7.33) and appropriate juggling of Eqs. (7.33) and (7.37)

(7.38)

$$+$$

It is clear that currents can be similarly represented.

The student will learn applications of the methods of symmetrical components in advanced courses in electric-circuit theory or a-c machinery. The purpose of this section is to show how three unbalanced vectors can be represented by nine symmetrical components. Since the application of this method requires far more analysis than that presented above, this section concludes with a brief statement of some of the results that have been obtained by means of symmetrical components.

When a power system has a single point of unbalance, such as a grounded transmission-line conductor, the problem can be solved more quickly by symmetrical components than by mesh equations.

In power systems that are symmetrical, each piece of equipment has a particular impedance to currents of each symmetrical component; these impedances differ from each other; but emfs of a particular component produce currents in the symmetrical system that depend on the impedances corresponding to that component. Thus the splitting of one problem into several simpler problems may (1) simplify the work of the complete solution, and (2) suggest to the investigator ways of controlling parts of the system by means of components of the voltage or current in that part of the circuit.

For further introductory discussion of symmetrical components, see reference,[4] Introduction and Chaps. I and II.

References

1. *Theory of Alternating-current Machinery;* A. S. Langsdorf; Chap. V, pp. 272–411, McGraw-Hill Book Company, Inc.; New York, 1937.
2. *Ibid.,* Chap. VI, pp. 412–476.
3. *Method of Symmetrical Coordinates Applied to the Solution of Polyphase Network;* C. L. Fortescue; *Transactions of the A.I.E.E.,* **37**, 1047, 1918.
4. *Symmetrical Components;* C. F. Wagner and R. D. Evans; McGraw-Hill Book Company, Inc., New York, 1933.
5. *Alternating Current Circuits;* J. M. Bryant, J. A. Correll, and E. W. Johnson; 3d ed., Chap. X, pp. 277–302; McGraw-Hill Book Company, Inc., New York, 1939.

Problems

7.01. Show that the power input to the load in a balanced three-phase system can be measured by means of one wattmeter.

7.02. A 25-cycle substation supplies a two-phase load having a power factor of 0.90 (current lagging). The substation is connected to a generating station by means of a three-wire line 50 miles long; each wire of the transmission line has an impedance $(0.27 + j0.30)$ ohm per mile. The phase voltages at the generating station are 66,000 volts and the load supplied by the generating station is 20,000 kw. Calculate: (*a*) the line currents; (*b*) the phase voltages at the substation; (*c*) the power factor at the generating station; (*d*) the power output of each phase at the generator; (*e*) the voltage between "outside" wires at the substation; (*f*) the loss in the transmission line.

7.03. Recalculate Problem 7.02 for a three-phase generator, substation, and load.

7.04. Two equal impedances are connected in series between two of the terminals of a Y-connected 220-volt generator. These impedances are very high compared to the generator impedances. What is the rms voltage between the junction point of the impedances and the neutral point of the generator? What is the rms voltage between this junction point and the third terminal of the generator?

7.05. A 500-volt, 25-cycle, 100-kva, three-phase, Δ-connected generator supplies full load to a balanced three-phase circuit whose power factor is 0.85. Calculate the phase currents in the generator. If the circuit of one phase of the generator were broken, would the generator still supply current to the load? If so, would the line currents and line voltages remain balanced? If the open-circuited phase does not prevent the generator from delivering power, how much power could it deliver without overloading the machine? Under the conditions of maximum load with one phase open-circuited, calculate the load supplied by each phase winding and the power factor of each of these loads.

7.06. The coils A, B of one transformer, and the coils C, D of a second transformer are interconnected as shown. Calculate the voltages from A_2 to B_2, from B_2 to C_2, and from C_2 to A_2 when a balanced two-phase voltage is connected to A_1, B_1, C_1. Calculate the voltages from A_1 to B_1 and from B_1 to C_1 when a balanced three-phase voltage is connected to A_2, B_2, C_2.
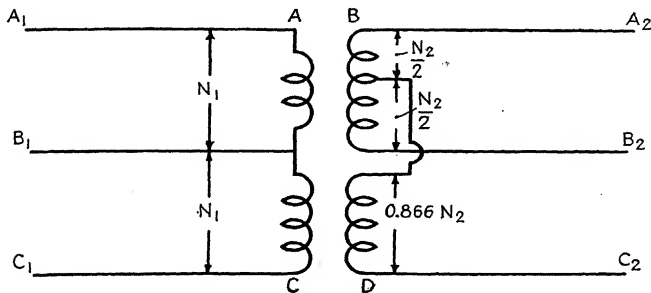
PROB. 7.06.

7.07. If two wattmeters are connected to measure a balanced three-phase load so that the sum of their readings $(P_A + P_B)$ is equal to the total load, show that the power factor angle θ is

7.08. A three-phase Y-connected generator supplies power to a balanced load at a line voltage of 220 volts. The power is measured by the two-wattmeter method; each wattmeter reads 10 kw. If each phase of the generator has an impedance of $(0.1 + j0.3)$ ohm, calculate the emf generated in each phase.

7.09. Each phase of a balanced, Δ-connected load is formed by an impedance in series with a resistance of 15 ohms. The voltage across the impedance in each phase is 80 volts; the voltage across the resistance in each phase is 150 volts; the phase voltage is 200 volts. If two wattmeters are connected to read the power input to the load, what powers will they indicate?

7.10. A Δ-connected generator not connected to a load produces phase emfs e

$$e = 18{,}600(\sin 2\pi 60t + 0.01 \sin 6\pi 60t + 0.0001 \sin 10\pi 60t) \qquad \text{volts}$$

Each phase resistance is 4 ohms and each phase inductance is 11 millihenrys. What is the rms value of the circulating current in the Δ? What power is dissipated in the Δ?

7.11. The line currents in a three-wire, three-phase system are $I_a = 60 + j40$, $I_b = -80 + j20$, $I_c = 20 - j60$. Calculate the symmetrical components of these currents.

7.12. The symmetrical components of a system of three-phase currents are $I_{a0} = 25\underline{/-40°}$, $I_{a1} = 35\underline{/20°}$, $I_{a2} = 20\underline{/80°}$. Synthesize the currents,

CHAPTER VIII

FILTERS AND TRANSMISSION LINES

8.01. Filter Networks; Introduction.—The a-c circuit theory presented in Chaps. V–VII, although generally applicable, has been discussed chiefly from the point of view of electric-power circuits. In such circuits the alternating emfs and currents have a single frequency—usually 25 or 60 cycles per second—although harmonic frequencies may be present and in certain circumstances they may contribute to phenomena of practical importance.

In communication circuits the currents may contain components of many frequencies. Thus the currents in a telephone circuit may have frequencies from 200 to 3,000 cycles per second corresponding to sound waves[1] of the same frequencies. The frequencies of currents in various parts of a radio or television transmitter or receiver make up the *audio-frequency spectrum* from about 30 cycles per second to several thousand cycles per second and the *radio-frequency spectrum* from about 30 kilocycles per second to 300 megacycles per second or more.*

The resistance of a conductor varies with frequency. As the frequency increases, the current density near the periphery of a cylindrical conductor increases, while the current density near the axis decreases. This *skin effect* is discussed in Chap. XIII. The reactance of a coil increases directly with frequency while the reactance of a condenser varies inversely with the frequency. Thus the magnitudes of the currents and voltages in a network of these elements vary with the frequency of the applied emf. Certain of these networks have important practical applications; they are called *filter networks* or *filters*.

The discussion of filters presented in this chapter is confined to a few simple examples. Thus only *passive* networks—those

* Phenomena at frequencies greater than about 50 megacycles are being intensively investigated. Some information[2,3] about them is now available; no doubt much more will be available when there is no longer the need for secrecy that now exists because of the war.

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in which there is no emf in the filter—are considered.   Simple
basic networks or *sections* such as those shown in Fig. 8.01 are
discussed.   It is shown that, if certain effects are produced
by inserting one such section in a circuit, then the effects are
enhanced by inserting several sections connected as shown in
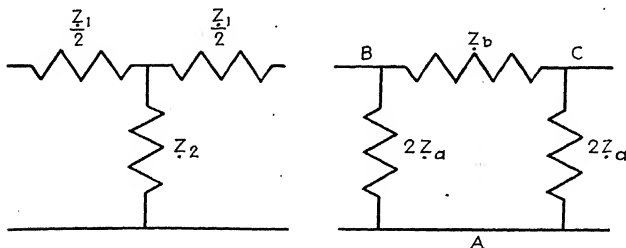
FIG. 8.01.

Fig. 8.02.   Methods are developed for calculating these effects
from the quantitative point of view.   Incidentally the circuit
parameters R, L, C are assumed throughout this chapter to be
constant.   In many cases these parameters vary appreciably
with frequency, but these complicated aspects of filter theory are
not discussed here.
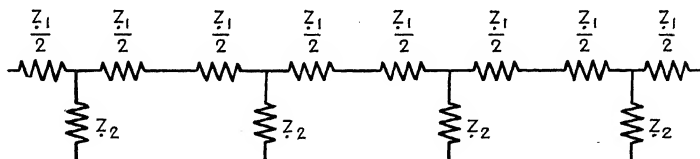
FIG. 8.02.

The filter section shown in Fig. 8.01, *left,* is called a symmetrical
T section; that in Fig. 8.01, *right,* a symmetrical π *section.*   The
networks shown in Figs. 8.01 and 8.02 are called passive *quadri-
poles, i.e.,* passive four-terminal networks.   The following
examples of the properties of filters are presented as a qualitative
basis for the analyses that are presented in subsequent sections.

Direct current that is required for the operation of thermionic
amplifiers (Chap. XV) is often obtained by connecting a source
of alternating current to a *rectifier,* which is essentially a device
that permits current flow through it in one sense but not in the

other. Thus the output of the rectifier comprises a series of half cycles of sine waves *all in one sense*. Such a current or voltage is called *pulsating*. A pulsating current is equivalent (Chap. IX) to alternating currents of many frequencies superimposed upon each other and upon a direct current. Therefore, to transform a pulsating current into a direct current, a filter is required having high series impedance and low shunt impedance to attenuate greatly the a-c components, and low series resistance and high shunt resistance to attenuate as little as possible the d-c component. A filter of T sections such as that shown in Fig. 8.02 fulfills these requirements if $Z_1/2$ represents a relatively high-inductance, low-resistance inductor and Z_2 represents a relatively high-capacitance, low-leakage condenser.

Thermionic amplifiers are often interconnected or *coupled* by networks having the properties of filters, although they are usually called *coupling circuits* rather than filters. An example is the case in which an a-c emf applied to terminals A, B is to be reproduced with as little attenuation as possible between terminals A, C, while B and C are to be isolated from each other for direct current, and low-resistance paths are to be provided from A to B and from A to C. The result can be attained by means of the π section shown in Fig. 8.01, *right*, if $2Z_a$ represents a relatively high-inductance, low-resistance inductor or *choke coil* while Z_b represents a relatively high-capacitance, low-leakage *by-pass condenser*.

Quadripoles often are used to limit the range of frequencies transmitted to a device. Thus it may be useful in order to decrease noise and to reduce interference to limit the frequency spectrum of the audio-frequency system in a transoceanic radio telephone transmitter to the range 100–3,000 cycles per second. This is accomplished by inserting a *band-pass filter* between the microphone and the apparatus to which it supplies energy. The filter is designed to attenuate as much as possible all frequencies less than 100 cycles and greater than 3,000 cycles, and to attenuate as little as possible frequencies in the range 100–3,000 cycles per second.

8.02. Filter Networks; Circuit Theory.—The discussion that follows is given in terms of symmetrical T sections. The student should reproduce the developments in terms of π sections for practice. The corresponding analyses for asymmetrical sections

are more complicated, but they involve no other fundamental principles than those presented below.
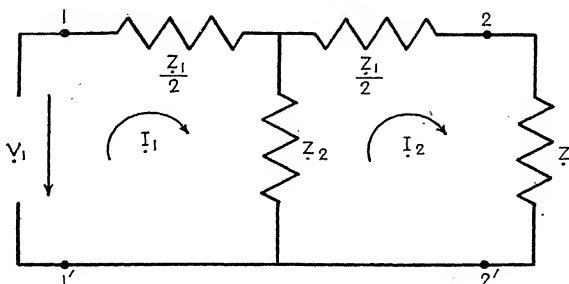
Fig. 8.03.

A single symmetrical T section will first be considered. The mesh equations and their solutions for the circuit shown in Fig. 8.03 are

$$V_1 = \left(\frac{Z_1}{2} + Z_2\right) I_1 - Z_2 I_2$$

(8.01)
$$0 = -Z_2 I_1 + \left(\frac{Z_1}{2} + Z_2 + Z\right) I_2$$

(8.02)
$$I_1 = \frac{\left(\dfrac{Z_1}{2} + Z_2 + Z\right) V_1}{\left(\dfrac{Z_1}{2} + Z_2\right)\left(\dfrac{Z_1}{2} + Z_2 + Z\right) - Z_2^2}$$

(8.03)
$$I_2 = \frac{Z_2 V_1}{\left(\dfrac{Z_1}{2} + Z_2\right)\left(\dfrac{Z_1}{2} + Z_2 + Z\right) - Z_2^2}$$

The *input impedance* is

(8.04)
$$\frac{V_1}{I_1} = \frac{Z_1}{2} + Z_2 - \frac{Z_2^2}{\dfrac{Z_1}{2} + Z_2 + Z}$$

Now consider several special cases. First, call Z_s the input impedance when $Z = 0$. From Eq. (8.04)

(8.05)
$$Z_s = \frac{\dfrac{Z_1}{2}\left(\dfrac{Z_1}{2} + 2Z_2\right)}{\dfrac{Z_1}{2} + Z_2}$$

Next, call Z_0 the input impedance when $Z = \infty$, *i.e.*, the output is open-circuited.   From Eq. (8.04)

$$(8.06) \qquad Z_0 = \frac{Z_1}{2} + Z_2$$

Finally, find the particular value of $Z = Z_k$ such that the input impedance and the load impedance are equal.   From (8.04)

$$Z_k = \frac{Z_1}{2} + Z_2 - \frac{Z_2^2}{\frac{Z_1}{2} + Z_2 + Z_k}$$

$$(8.07) \qquad Z_k = \sqrt{\frac{Z_1}{2}\left(\frac{Z_1}{2} + 2Z_2\right)} = \sqrt{Z_0 Z_s}$$

The quantities defined by (8.05), (8.06), (8.07) are called respectively the *short-circuit input impedance*, the *open-circuit input impedance*, and the *input iterative impedance*.   From Eqs. (8.02), (8.03), and (8.07) it follows that

$$(8.08) \qquad \frac{I_1}{I_2} = \frac{\frac{Z_1}{2} + Z_2 + \sqrt{\frac{Z_1}{2}\left(\frac{Z_1}{2} + 2Z_2\right)}}{Z_2}$$

if the circuit is terminated by its input iterative impedance.   If this iterative impedance is replaced by a second section, which, in turn is connected to Z_k, the ratio of the input current to the output current is $\left(\frac{I_1}{I_2}\right)^2$.   It follows that for n sections the ratio of input to output current is $\left(\frac{I_1}{I_2}\right)^n$.   Therefore it is useful to define a complex quantity γ known as the *transfer constant** in the following:

$$(8.09)\dagger \qquad \varepsilon^\gamma = \frac{\frac{Z_1}{2} + Z_2 + \sqrt{\frac{Z_1}{2}\left(\frac{Z_1}{2} + 2Z_2\right)}}{Z_2}$$

* It is not constant; γ is a complex number that varies with frequency.

† For convenience in calculation, this definition is often stated in the following form:

$$\varepsilon^\gamma = \frac{\sqrt{1 + \frac{Z_1}{4Z_2}} + \sqrt{\frac{Z_1}{4Z_2}}}{\sqrt{1 + \frac{Z_1}{4Z_2}} - \sqrt{\frac{Z_1}{4Z_2}}}$$

which, the student should show, is identical to Eq. (8.09).

This definition can be combined with the relations of Eqs. (8.05) to (8.07) to show that

$$(8.10) \qquad \tanh \gamma = \sqrt{\frac{Z_s}{Z_0}}$$

Thus from measurements of Z_s and Z_0 for one section of a filter, at one frequency, the magnitude and phase of the ratio of the input current to the output current can be calculated for an n-section filter terminated by the image impedance, *for that frequency:*

$$(8.11) \qquad \frac{I_1}{I_{n+1}} = \varepsilon^{n\gamma}$$

Calling V_{n+1} the impedance drop in the load it can be shown also that

$$(8.12) \qquad \frac{V_1}{V_{n+1}} = \varepsilon^{n\gamma}$$

and therefore

$$(8.13) \qquad \frac{V_1 I_1}{V_{n+1} I_{n+1}} = \varepsilon^{2n\gamma}$$

The transfer constant or factor γ is in general a complex number. Its real part (putting $\gamma = \alpha + j\beta$) is called the *attenuation constant* or *factor* and the imaginary part β is called the *phase constant* or *factor*. Therefore

$$(8.14) \qquad \frac{I_1}{I_{n+1}} \underline{/\phi_1 - \phi_{n+1}} = \varepsilon^{n\alpha} \underline{/n\beta}$$

so that the magnitude of the ratio of the input current to the output current is an exponential function. This result leads to the definition of a convenient unit of measure of current, voltage, and power ratios. The unit is called the *decibel** (db), and it is defined as follows. The number of decibels, loss or gain, is calculated from the ratio $\dfrac{P_a}{P_b}$, *always greater than* 1. The number of decibels is calculated from

$$(8.15) \qquad 10 \log_{10} \frac{P_a}{P_b} \qquad\qquad \text{decibels}$$

* A more logical measure of gain or loss is the *natural* logarithm of the power or volt-ampere ratio. The unit is called the *neper*. Its use is increasing, but at the moment the decibel appears to be leading the field.

For the special case discussed above, *i.e.*, when the input and output impedances are both equal to Z_k, the number of decibels calculated from Eq. (8.15) is the same as the number of decibels calculated from

(8.16)   decibels

or

(8.17) $20 \log_{10} \cdot$ decibels

It should be emphasized that Eqs. (8.16) and (8.17) are equivalent to Eq. (8.15) only when the input and output impedances are equal to Z_k. If the pair of terminals a are input terminals of a device having output terminals b, the result calculated from Eq. (8.15) is called a *transfer power loss*. If the pair of terminals b are input terminals and a are output terminals then the result of Eq. (8.15) is called the *transfer power gain*. The result of calculating Eq. (8.16) or (8.17) for the first case is called voltage or current *attenuation;* in the latter case it is often called the voltage or current *amplification.*

It is clear that the insertion of a *passive* network in a circuit always produces a transfer *power* loss, because there is always some dissipation of heat in such a network. The usefulness of the decibel as a unit of measurement is illustrated by the following simple example. Suppose that the attenuation constant $\alpha = 1.15$ for a particular T sec ion is known and it is desired to obtain, from a particular source of current I_1, currents of magnitude approximately equal to 60 decibels and 140 decibels lower than I_1 in a load impedance equal to the iterative impedance of the section. Accordingly n must be chosen [Eq. (8.14)] so that

$$20 \log_{10} \frac{I_1}{I_{n+1}} = 60 = 20 n_1 \alpha \log_{10} \varepsilon$$

and

$$20 \log_{10} \frac{I_1}{I_{n+1}} = 140 = 20 n_2 \alpha \log_{10} \varepsilon$$

so that $n_1 = 6$ and $n_2 = 14$. Thus 14 sections are required, and, when they are all connected, the output current is one tenmillionth of the input current. If 6 sections are used the output

current will be one one-thousandth of the input current. The decibel losses are additive

Loss of 6 sections.................................. 60 db
Loss of 8 sections.................................. 80 db
Loss of 14 sections................................ 140 db

The greatest advantage of this system is demonstrated when losses (−decibels) in filters and gains (+decibels) in thermionic amplifiers, both expressed in decibels, are added algebraically to calculate the over-all result.

A pair of impedances Z_{i1} and Z_{i2} connected respectively in the input and output circuits of Fig. 8.03 and having values such that the impedances looking either way from terminals 1, 1′ are equal, and the impedances looking either way from terminals 2, 2′ are equal, are called the *image impedances* of the network. For the particular simple case considered here the iterative impedance, Eq. (8.07), is an image impedance for both input and output circuits.

8.03. Filter Networks; Band-pass Properties.—The properties of a series of symmetrical T sections of special form are analyzed briefly in this section. More complicated examples are omitted[4] in the interests of simplicity and clarity.

Assume that a series of identical symmetrical T sections is to be terminated by the iterative impedance Z_k of one section. The problem to be considered is to find the general properties of the series $\left(\dfrac{Z_1}{2}\right)$ elements and of the shunt (Z_2) elements such that the attenuation constant (α) is zero.

If $\alpha = 0$ then

$$(8.18) \qquad \gamma = j\beta$$

and, from Eq. (8.10)

$$(8.19) \qquad \tanh \gamma = \tanh j\beta = j \tan \beta = \sqrt{\frac{Z_s}{Z_0}}$$

It follows therefore that $\sqrt{\dfrac{Z_s}{Z_0}}$ must be a pure imaginary for α to be zero. Substituting for Z_s and Z_0 from Eqs. (8.04) and (8.05), this means that

$$(8.20) \qquad j \tan \beta = \sqrt{\frac{\dfrac{Z_1}{2}\left(\dfrac{Z_1}{2} + 2Z_2\right)}{\left(\dfrac{Z_1}{2} + Z_2\right)^2}}$$

must be imaginary. Put

$$(8.21) \qquad \rho =$$

Then, substituting in Eq. (8.20)

$$(8.22) \qquad j \tan \beta = \sqrt{\frac{\rho(\rho + 1)}{(\rho + \frac{1}{2})^2}}$$

The right-hand member is imaginary if ρ is real and

$$(8.23) \qquad -1 < \rho < 0$$

Therefore all frequencies for which $\dfrac{Z_1}{4Z_2}$ *is a real number between* -1 *and* 0 *are passed by the filter without attenuation.* All frequencies outside of this pass-band are attenuated—more or less depending upon the magnitude of $\dfrac{Z_1}{4Z_2}$ and upon the number of sections in the filter.

The quantity ρ is real and between 0 and -1 only if Z_1 and Z_2 are imaginary numbers. The iterative impedance Z_k can be written, from Eqs. (8.07) and (8.21),

$$(8.24) \qquad Z_k = 2Z$$

Within the pass-band the square root is imaginary [see Eq. (8.22)]. Since Z_2 is also imaginary, it follows that Z_k is real, *i.e.*, the iterative impedance is a resistance. Note that the sign $(+$ or $-)$ of $\rho(\rho + 1)^{\frac{1}{2}}$ is so chosen that Z_k is a positive real number.

The filter therefore comprises T sections of which the elements are reactances. Although there are no inductors having zero resistance and no condensers having zero leakage, this ideal condition can be approached in practice. Since the analysis of a *non-dissipative* network (*i.e.*, one containing no resistance) is simpler than the analysis of a dissipative network, a preliminary survey of a filter problem in the form given in this section is often practically useful.

A more detailed analysis of the voltage and current at various parts of a series of T or π sections is given in one[4] of the references listed at the end of the chapter.   This aspect of the filter problem is not presented here because its essential results are contained in the theoretical analysis of the properties of two-wire transmission lines to which the remainder of this chapter is devoted.

8.04. Transmission Lines; Differential Equations and Steady-state Solutions.—The name *transmission line* is a generic name for a system of conductors used to transmit electric energy. Thus a three-phase 66,000-volt open-wire transmission line consists of three relatively heavy conductors (of the order of No. 00 A.W.G. copper wire), separated from each other by a distance of a meter or more and suspended from insulators that are attached to steel towers several hundred feet apart. An analysis of the steady-state properties of the line requires in this case a knowledge of the capacitance per unit length from one wire to another and from one wire to ground, the inductance per unit length, the resistance per unit length, the leakage current in the insulators, the temperature of the wire from point to point, the losses produced by the ionization of the air near the conductors (*corona loss*), as well as the dimensions of the line and the nature of the emfs applied to it.   In a *cable*, the parallel conductors are separated by a solid insulator.   Thus if a variable emf is applied to the terminals of a cable, the inductive drop, resistance drop, charging current, and leakage current vary continuously with distance along the cable.   It is said in these cases that the parameters (inductance, conductance, etc.) are *continuously and uniformly distributed* along the line.   Finally the applied emfs may not be sinusoidal and the transients may be important.   The general problem is certainly not easy.

In the solution of many transmission-line problems two kinds of approximations can often be made.   (1) It may be assumed that a network of one or several T sections is approximately equivalent to the line under consideration.   This amounts to assuming that the distributed parameters are *lumped, i.e.,* they are representable by one or several inductors, resistors, and capacitors.   (2) It may be assumed that the effect of one or more parameters is negligible.   Thus in Sec. 7.02 it was assumed that the distributed inductance and the distributed resistance could be lumped into a single line impedance $Z = R + j\omega L$,

and that the distributed capacitance and the conductance from one wire to another were negligibly small. The accuracy of such approximations clearly varies with the conditions of the problem; there are no definite rules to follow. If these approximations are judged to be sufficiently accurate, the problems can be solved as if the line were a filter by the methods described in Secs. 8.01 to 8.03.

The problem of a two-wire transmission line with uniformly distributed parameters is attacked below by the method first used in Sec. 2.05 to obtain the steady-state solution for a d-c line. The differential equations for a transmission line with parameters R, L, C, G per unit length are derived, together with their steady-state solutions. Note that the same result can be obtained by assuming that the line consists of an infinite number of symmetrical T sections, each representing an infinitesimal length of the line.[5] The circuit parameters are assumed to be constant. They are in general both measurable and calculable. Methods for calculating R and G are presented in Chap. III; for L in Chap. XIII; and for C in Chap. XI. These parameters are

R, ohms per meter—twice the resistance of one meter of one conductor.
L, henrys per meter—inductance per unit length.[*]
G, mhos per meter—leakage conductance per unit length.
C, farads per meter—capacitance between the two conductors, per unit length.

The line is shown diagrammatically in Fig. 8.04. If the instantaneous voltage is v at a point x meters from the terminals 2, 2′, it is

$$(8.25) \qquad \frac{\partial v}{\partial x}$$

at a point $x + dx$ meters from 2, 2′, from which

$$(8.26) \qquad \frac{\partial v}{\partial x} \qquad \qquad \frac{\partial i}{}$$

---

[*] For present purposes it is sufficient to note that there is an instantaneous voltage drop per unit length of line proportional to the rate of change of current with time; L is the factor of proportionality between *this* part of the drop and the time rate of change of current. A more comprehensive definition of L is worked out in Chap. XIII.

If the instantaneous current is i at a point x meters from the terminals 2, 2′, it is

(8.27)

at a point $x + dx$ meters from 2, 2′ so that
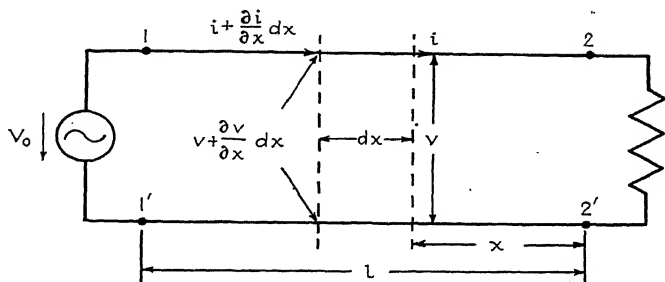
FIG. 8.04.

The line voltage v and the line current i are functions of x and t. The partial derivatives in Eqs. (8.25), (8.26), (8.27), (8.28) are symbols for the following:[6]

$$(8.29) \quad \begin{aligned} \frac{\partial v}{\partial x} &\underset{\Delta x \to 0}{} \frac{+\ \Delta x),\ t] - v(x,\ t)}{\Delta x} \\ \frac{\partial v}{\partial v} & \quad v[x,\ (t + \Delta t)] - v(x,\ t) \end{aligned}$$

And similarly for $\partial i/\partial x$ and $\partial i/\partial t$. No additional theory of partial differential equations is required here because the solving of Eqs. (8.26) and (8.28) is based upon the operations defined by Eq. (8.29) and upon the inverse operation—an analytical expression for $\partial i/\partial x$ is integrated by following the usual rules of integration, *assuming t to be constant.*

Since no attempt will be made here to investigate the transient effects in transmission lines, it will now be assumed that the applied voltage, and therefore v and i all along the line, are sinusoidal. The steady-state solutions of Eqs. (8.26) and (8.28) can be obtained conveniently by the method described at the

end of Sec. 6.05. Substitute complex numbers V and I for v and i, and substitute $j\omega$] $\qquad$ for $\dfrac{\partial v}{\partial t}$ and $\dfrac{\partial i}{\partial t}$:

(8.30)

(8.31)

Put

(8.32) $\qquad Z = R + j\omega L$ (the line impedance per unit length)
(8.33) $\qquad Y = G + j\omega C$ (the line admittance per unit length)

and substitute in Eqs. (8.30) and (8.31). Take the derivative with respect to x of both sides of the resulting equations; there results

(8.34)

(8.35)

The solution of Eq. (8.34) for the voltage V the distance x from the receiving end is

(8.36) $\qquad\qquad V =$

in which

(8.37) $\qquad\qquad \gamma =$

The complex number γ is the propagation constant per unit length; α is the attenuation constant per unit length; and β is called the wave length or phase-shift constant per unit length of line. Note the similarity of these quantities to those defined in Sec. (8.02) for a filter.

The current at the same distance x from the receiving end is obtained by substituting Eq. (8.36) in Eq. (8.30). The result is

(8.38) $\qquad\qquad = \dfrac{1}{Z_k}\left(A\varepsilon^{\gamma x} - B\varepsilon^{-\gamma x}\right)$

in which

(8.39)

is called the *characteristic impedance* of the line.

The number $A\varepsilon^{\gamma x}$ is a component of the vector voltage V and is itself therefore a vector voltage. Designate this component of the vector voltage V by the symbol V_i, *viz.*,

$$(8.40) \qquad\qquad V_i = A\varepsilon^{\gamma x}$$

The cosine function that represents the instantaneous value of this voltage is

$$(8.41) \qquad\qquad v_i = \sqrt{2}\; V_i \cos\,(\omega t + \phi)$$

where V_i is the modulus of V_i and ϕ is the phase angle of V_i. Since $\gamma = \alpha + j\beta$ and since in general A has a phase angle, which may be designated θ_a, Eq. (8.41) becomes

$$(8.42) \qquad\qquad v_i = \sqrt{2}\; A\varepsilon^{\alpha x} \cos\,(\omega t + \beta x + \theta_a)$$

This component of the resultant voltage is a traveling wave (see page 208) that travels with a velocity $\dfrac{\omega}{\beta}$ in the direction *toward* the receiver (terminals 22′) and that at the receiver has an amplitude A and phase angle θ_a and that at the sending end (terminals 11′) has an amplitude $A\varepsilon^{\alpha l}$ and a phase angle $(\theta_a + \beta l)$. This wave is called the *incident* voltage wave.

In exactly the same manner the number $B\varepsilon^{-\gamma x}$ corresponds to an instantaneous voltage

$$(8.43) \qquad\qquad v_r = \sqrt{2}\; B\varepsilon^{-\alpha x} \cos\,(\omega t - \beta x + \theta_b)$$

This component of the resultant voltage is a traveling wave that travels from the receiver toward the sending end with the same velocity as the incident voltage wave and that at the receiver has an amplitude B and phase angle θ_b and that at the sending end has an amplitude $B\varepsilon^{-\alpha l}$ and a phase angle $(\theta_b - \beta l)$. This wave is called the *reflected* voltage wave.

The resultant voltage at any point on the line, at any time t, is then

$$(8.44) \qquad\qquad v = v_i + v_r$$

where v_i and v_r are given by Eqs. (8.42) and (8.43). The rms value of this voltage is the modulus of the sum

$$(8.45) \qquad V = A\varepsilon^{\alpha x}\underline{/\theta_a + \beta x} + B\varepsilon^{-\alpha x}\underline{/\theta_b - \beta x}$$

and its phase angle is the phase angle of this sum.

The resultant current can be analyzed in the same manner, *viz.*, an incident current wave

$$(8.46) \qquad i_i = \frac{\sqrt{2}}{Z_k} A \varepsilon^{\alpha x} \cos\ (\omega t + \beta x + \theta_a - \theta_k)$$

and a reflected wave

$$(8.47) \qquad i_r = - \sqrt{2} \qquad \cos\ (\omega t - \qquad - \theta_k)$$

The rms value of the resultant current is the modulus of the sum

$$(8.48) \quad I = \qquad\qquad - \theta_k - \qquad\qquad \underline{- \beta x - \theta_k})$$

and its phase angle is the phase angle of this sum.

8.05. Transmission Lines; Special Cases.—The concept of traveling waves discussed in Sec. 8.04 can be clarified by consideration of the special case of a transmission line for which $R = G = 0$.   It follows that, *for this lossless line,*

$$(8.49) \qquad\qquad \gamma = \sqrt{ZY} = j\omega \sqrt{LC} = j\beta$$

and

$$(8.50) \qquad\qquad Z_k =$$

Thus the propagation constant of a lossless line is imaginary, and the characteristic impedance is a pure resistance.   Therefore the current and voltage at any point distant x from the receiving-end terminals 22' at any instant t are, from Eqs. (8.42), (8.43), (8.46), and (8.47)

$$(8.51) \quad v = \qquad\qquad \cos \qquad\qquad \theta_a)$$
$$+ \sqrt{2}\, B \cos\ (\omega t - \beta x + \theta_b)$$

$$(8.52) \quad i = \sqrt{2}\, \sqrt{\frac{C}{L}}\, A \cos\ (\omega t + \beta x + \theta_a)$$
$$\frac{\sqrt{C}}{L}\, B \cos\ (\omega t - \beta x + \theta_b)$$

Each of the components of v, Eq. (8.51), is analyzed in the following paragraph.

Consider first the voltage, Eq. (8.51). It is instructive as noted in Sec. 8.04 to analyze the two terms separately. Note that $\cos(\omega t - \beta x + \theta_b)$ represents the variation of voltage with x and t *for the second term*, which is designated v_r. *At a particular point* in the line x meters from the receiving end the contribution of the second term to the voltage varies sinusoidally *with time.* Its frequency f and period $T = 1/f$ are equal to those of the applied emf. *At a particular instant of time t* the contribution of
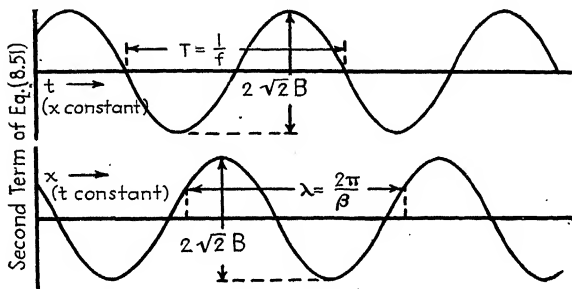
Fig. 8.05.

the second term to the voltage varies sinusoidally *with x.* The distance $x_2 - x_1$ from x_1, where the voltage is v_{r1} and v_r is increasing with x at a given instant, to the point x_2, which is the point nearest x_1 where the same conditions are satisfied at the same instant, is called the *wave length* λ. Figure 8.05 illustrates these points. It follows from the diagram that if, at a point x_1, the contribution of the second term of Eq. (8.51) to the voltage is zero at time t_1, the voltage v_r at some other point x_2 *less than* $\frac{\lambda}{4}$ *meters from x_1* is zero at some other time t_2. These two events are represented by

$$\omega t_1 - \beta x_1 = \frac{\pi}{2}$$

(8.53)

$$\omega t_2 - \beta x_2 = \frac{\pi}{2} \qquad \left(\cos\frac{\pi}{2} = 0\right)$$

or

$$\omega t_1 - \beta x_1 = \omega t_2 - \beta x_2$$

$$(8.54) \qquad \frac{x_2}{t_2 - t_1} \qquad \frac{\omega}{\beta} \qquad \text{meters per second}$$

Thus the wave v_r appears to travel along the line *from the receiving end to the sending end* with a velocity ω/β. This velocity is called the *phase velocity*.*

The first term v_i of Eq. (8.51) has properties similar to those of v_r except that v_i appears to travel *from the sending end* with a velocity ω/β. Thus v_i is said to represent the *incident wave* and v_r the *reflected wave*, both with respect to the receiving end of the line.

The comments of the last two paragraphs apply also to the terms of the current, Eq. (8.52).

The constants A and B are determined below for the special case when the line is terminated at its receiving end by its *characteristic resistance R_k*

$$(8.55) \qquad R_k = \sqrt{\frac{L}{C}} \qquad \text{ohms}$$

In this special case Eqs. (8.51) and (8.52) must combine for $x = 0$ to satisfy

$$(8.56) \qquad R_k i_0 =$$

for any instant of time. The relation in Eq. (8.56) is satisfied if $B = 0$. Thus in the case of a non-dissipative line ($R = G = 0$) terminated by its characteristic resistance R_k *there is no reflected wave*. The final results in terms of the rms voltage and current at any point along the line are

$$(8.57) \qquad V = V_0 \cos \beta x \qquad \text{rms volts}$$

$$(8.58) \qquad I = \frac{V_0}{R_k} \cos \beta x \qquad \text{rms volts}$$

where V_0 is the rms voltage between the output terminals.

* For the lossless line here considered the phase velocity is equal to the velocity of light in a vacuum.[6] The phase velocity in general is *not* equal to the velocity of a pulse or signal, which according to contemporary experiment and theory does not exceed the velocity of light (Chap. XV). The phase velocity, on the other hand, can exceed the velocity of light; in particular, the phase velocity in a *wave guide*[9] may exceed in magnitude the velocity of light.

Note that the impedance $\dfrac{V}{I}$ at any point (including input and output) is

$$(8.59) \qquad\qquad \frac{V}{I} = R_k \qquad\qquad \text{ohms}$$

The more general problems that can be developed from the relations presented in Sec. 8.04 are omitted in the interest of simplicity. The algebra involved in such problems is a little tedious.[5] Further information concerning special cases can be obtained by solving problems at the end of this chapter.

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1. *Sound;* A. T. Jones; D. Van Nostrand Company, Inc., New York, 1937.
2. *Ultra-high-frequency Techniques;* J. G. Brainerd (ed.), Glenn Koehler, H. J. Reich, L. F. Woodruff; D. Van Nostrand Company, Inc., New York, 1942.
3. *Microwave Transmission;* J. C. Slater; McGraw-Hill Book Company, Inc., New York, 1942.
4. *High-frequency Alternating Currents;* Knox McIlwain and J. G. Brainerd; 2d ed., Chap. IX, pp. 280–325; John Wiley & Sons, Inc., New York, 1939.
5. *Ibid.;* Chap. X, pp. 326–351.
6. *Advanced Calculus;* E. B. Wilson; pp. 93–148; Ginn and Company, Boston, 1912.
7. *Principles of Transmission in Telephony;* M. P. Weinbach; Chap. IV, pp. 25–42, The Macmillan Company, New York, 1924.
8, 9. Reference 2, pp. 359, 462–463 (footnote 13).

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Transmission Networks and Wave Filters; T. E. Shea; D. Van Nostrand Company, Inc., New York, 1929.
Principles of Electric Power Transmission; L. F. Woodruff; 2d ed., John Wiley & Sons, Inc., New York, 1938.

Problems

8.01. Determine the band or bands of frequencies that will be passed without attenuation by the section shown below.
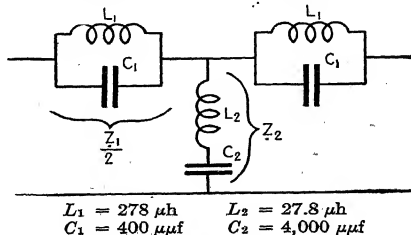
$L_1 = 278 \,\mu h$ $L_2 = 27.8 \,\mu h$
$C_1 = 400 \,\mu\mu f$ $C_2 = 4{,}000 \,\mu\mu f$
Prob. 8.01.

8.02. Repeat Problem 8.01, interchanging Z_1 and Z_2.

8.03. Plot attenuation (db) *vs.* frequency for frequencies outside of the pass-band (or bands) for the filter section of Problem 8.02, assuming that the section is terminated by a resistance equal to the characteristic resistance of the filter for

$$\omega = \frac{1}{\sqrt{L_1 C_1}}$$

8.04. Plot attenuation *vs.* frequency for the low-pass filter shown below (*left*), assuming that one section is terminated by the characteristic resistance of the section for $\omega = \frac{\omega_c}{5}$, where ω_c corresponds to the cutoff frequency.
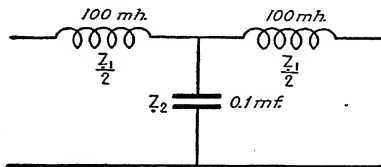
PROB. 8.04.

8.05. Using the low-pass filter of Problem 8.04 as a prototype, derive a T section such that: (1) $Z_1' = M Z_1$; and (2) $Z_k = Z_k'$.   Use $M = 0.6$.
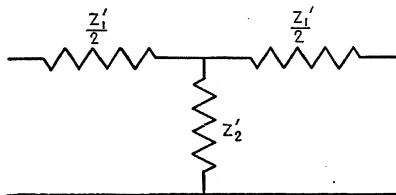
PROB. 8.05.

8.06. Connect the sections of Problems 8.04 and 8.05 together, terminate as described in Problem 8.04 and plot attenuation *vs.* frequency.   Compare with the curve for Problem 8.04.

8.07. Neglecting the series resistance and shunt conductance of a transmission line, plot the input impedance *vs.* the length l from $l = 0$ to $l = \lambda = \frac{2\pi}{\beta}$, assuming that the line is open-circuited at the far end.

8.08. Repeat Problem 8.07 for a short-circuited line.

8.09. Pairs of wires in commonly used telephone cables are so chosen that the series resistance greatly exceeds the series reactance and the shunt susceptance is large compared to the leakage conductance.   Show that, for such a cable pair

$$\alpha = \beta = \sqrt{\frac{\omega C R}{2}} \qquad \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{RC}} \qquad Z_c = \sqrt{\frac{R}{\omega C}}\, \varepsilon^{-j\frac{\pi}{4}}$$

CHAPTER IX

NON-LINEAR CIRCUITS

9.01. Introduction.—It is the purpose of this chapter to indicate a few of the problems of circuit theory that arise whenever a circuit element R, L, or C varies with the current through the element, or with the voltage between its terminals. Such circuits are called *non-linear circuits*.

Throughout Chaps. I to VIII it has been assumed that the circuit elements R, L, and C are constants. The differential equations relating the charges, currents, and applied voltages for circuits containing constant resistances, inductances, and capacitances were found to be linear. Two important complementary results were derived from these analyses: (1) the production of a steady-state current of the form $A \cos \omega t$ in a network requires, within the network, an emf of the same frequency $\omega/2\pi$, and (2) if a network contains emfs $A_1 \cos \omega_1 t$, $A_2 \cos \omega_2 t$, . . . , the equations representing the network may be solved separately for the currents produced by each emf, and the results of all such calculations may be added for each branch or mesh to obtain the total current (principle of superposition). Many circuits, consisting of constant circuit elements, are therefore *linear* in the sense described above, or at least the inaccuracies of the circuit currents, calculated by assuming that the elements are constant, are negligibly small.

There remain the circuits in which the variations of R, L, and C are important from the engineering point of view. No simple general method of solving them has been developed. Therefore each such problem is treated as a special case, and methods finally evolve, usually by cut-and-try procedures, for handling it and similar problems. The discussion of non-linear circuits is often omitted in elementary texts on electricity and electromagnetism, perhaps because of the lack of general methods of solving such problems. Yet the continuous development of the theory of electrical engineering and the application of the theory to the design and production of electrical devices reveal

an increasing number of non-linear circuits that must be solved, for economic reasons if for no other. Therefore it is useful at least to emphasize the existence of these problems in an introductory course in electricity and electromagnetism.

The distinguishing property of a non-linear circuit element is that its presence in an a-c circuit may cause the production of steady-state as well as transient currents having alternating components whose frequencies differ from those of the applied emfs. *This property in general corresponds to a resistance, inductance, or capacitance, which is a function of current.*

If the resistance, inductance, or capacitance is a function only of time, or only of the spatial position or change of position of the device it represents, it is *not* a non-linear circuit element according to the meaning of that term assumed throughout this chapter. If *R*, *L*, or *C* varies according to a known function of time or of spatial coordinates, the problem can in general be solved analytically by the methods discussed in previous chapters.

Non-linear circuit elements encountered in engineering problems include the following important examples:

1. Inductors having ferromagnetic cores. The electromagnetic effects produced in the winding or windings of such an inductor depend in general upon the magnitudes of the currents in the windings, the size, material, and arrangement of the core, and upon the previous magnetic history of the core. The self-inductances of the windings and the mutual inductances between pairs of them are not constant. They vary with the currents in the windings. Thus the accurate analysis of the commonly used two-winding transformer involves non-linear effects. Fortunately, practically satisfactory approximations can often be made in such a manner that an assembly of linear elements can be used as an *approximately equivalent circuit* to facilitate the solution of electric-circuit problems involving transformers.

2. Vacuum, and gaseous discharge, tubes. The graph of the voltage *v* applied to two electrodes of such a tube *vs.* the current *i* through the tube is usually a curve, which often has one or more inflection points. Thus if dv/di is called the *dynamic resistance* of such a device, this resistance is a function of the current through the device and is therefore non-linear. A non-linear resistance of this type is discussed in the following sections of this chapter (see also Chap. XV).

3. Substances such as thyrite. Thyrite,[1] a substance obtained by specially heat-treating a mixture of carbon and clay, has a voltage-current characteristic of such form that dv/di decreases rapidly as the voltage is increased. It is used in *lightning arresters*, connected from a transmission line to ground. At normal line voltage the current through the arrester is small (a few milliamperes), but, at the higher voltages produced by lightning, its "resistance" decreases and a large current flows momentarily through the arrester, thus causing the line to return to normal operating conditions before connected apparatus is damaged.*

4. Rectifiers.[2] These are devices for which the resistance is much higher for voltages applied in one sense to their terminals than for voltages applied in the opposite sense. The ideal rectifier would be one for which dv/di is infinite when one of its terminals is positive with respect to the other, and zero where the polarity is reversed. This ideal has not been attained in practice. Thermionic rectifiers (Chap. XV) and the contacts between dissimilar substances such as cuprous oxide and copper are examples of rectifiers.

5. Skin effect. The current density in a cylindrical conductor carrying a current parallel to its axis is usually uniform over the cross section of the wire for direct current and for low-frequency alternating current. The current density is greater near the periphery than it is near the axis for relatively high-frequency alternating currents. Therefore the resistance of a conductor increases with frequency. This problem is discussed in Chap. XIII (Sec. 13.06).

6. Corona. If the potential of a transmission wire with respect to the earth or with respect to other near-by conductors rises beyond a critical point, depending chiefly upon the size, shape, and relative positions of the conductors, the air surrounding the conductor becomes ionized. The effect is produced by the acceleration of free electrons in the air by the electric field in the

* It might be inferred from this qualitative explanation that the operation of this thyrite arrester is simple from the theoretical point of view. *It is not.* The complete analysis depends upon such factors as the electrical effects of the interaction of atmospheric electricity and the transmission line, the rate of propagation of transient effects along the line, the voltage-current characteristic of the arrester, and the stability of the entire generating and distributing system during the time interval under consideration.

space surrounding the conductors (Chap. X).   Energy is required to produce the ionization.   This energy is supplied by the electric circuit and is transformed by the mechanisms of the corona discharge into light and heat.   Therefore the resistance of the line, *i.e.*, the ratio of the instantaneous power input p to the square of the instantaneous current i^2, is not constant, but varies with the potential of the line with respect to surrounding conductors. Corona losses[7] in power lines are often large enough to be of practical importance.

The sections that follow consist of the analysis of a particular non-linear circuit problem from several points of view.   No detailed discussion of the many mathematical procedures that have been used to solve other problems involving non-linear circuit elements is included in this chapter.   These methods include the use of the *differential analyzer*,[3] a mechanical device for solving ordinary differential equations; the *step-by-step*[4] and *isocline*[5] methods for obtaining approximate solutions of equations whose integrals cannot be obtained by conventional analytical means; and the graphical methods that are commonly used in many fields.   Finally there is a very useful method that might not occur to the armchair electrical engineering student: the circuit, or an electrical or mechanical analogue of it, can be assembled in the laboratory and the solution can be obtained by means of an oscillograph or an analogous device.

9.02. A Particular Non-linear Circuit Element.—A thermionic rectifier is chosen arbitrarily as a particular example of a non-linear circuit element.   Some of its characteristics are described in this section, and a simple circuit problem involving the use of the rectifier is stated.   Although this problem happens to have little practical significance, its solution by several means in subsequent sections is simple and instructive.   The problem is chosen so that the student can understand clearly at the outset the physical phenomena that are involved, and then he can follow the several methods of solution without becoming lost in a maze of mathematical symbols.

The particular thermionic rectifier that is to be considered is shown in Fig. 9.01.   The arrangement of the parts of the tube is shown at the left and the symbol used to represent the tube is shown at the right.   The metal plate P is the anode of the tube and the thoriated tungsten filament C is the cathode.   The

filament is supported by wires embedded in the glass mass M, and the plate is suspended from a similar but smaller piece of glass at the top of the tube.   The electrodes are surrounded by a glass shell.   The wires, by which the electrical connections are made to the two ends of the filament and to the plate, extend through the glass walls of the tube.   The tube is evacuated and sealed.   The plate lead is soldered to the metal cap at the top of the tube.   The filament leads are soldered to two prongs at the bottom of the base B, after which the base is cemented to
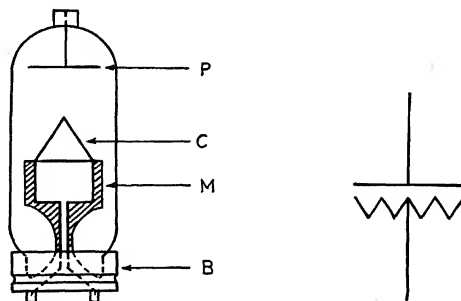
Fig. 9.01.

the bottom of the glass envelope of the tube.   The tube base fits into a standard socket to the terminals of which external connections to the filament can be made conveniently.   The use of the socket makes it possible to replace the tube quickly without unsoldering any connections.

The filament is designed to attain its prescribed operating temperature when the current through it is 3.25 amperes, or the voltage across its terminals is 10.0 volts.   The filament emits electrons when it is incandescent.   If the plate is positive with respect to the cathode, some of these thermionic electrons are attracted to the plate, *i.e.*, a current flows through the tube from plate to filament in the conventional sense.   If the plate is negative with respect to the cathode, the electrons are repelled by the plate and no current flows.

When the plate is positive with respect to the filament, the potential difference v is *not* proportional to the current i through the tube.   *The device is a non-linear resistance*.   The experimentally determined relation between v and i is shown in Fig.

9.02. Although this is the average *static-characteristic curve* obtained by testing many tubes and taking an average of all the data, it is assumed in the following sections that the curve describes the voltage-current characteristic of the particular tube that is the subject of the problem discussed therein.
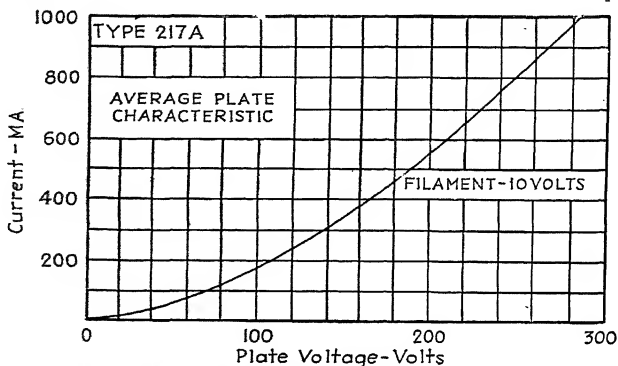
FIG. 9.02.—(*Reproduced by permission of the Radiotron Division of the RCA Manufacturing Co., Inc.*)
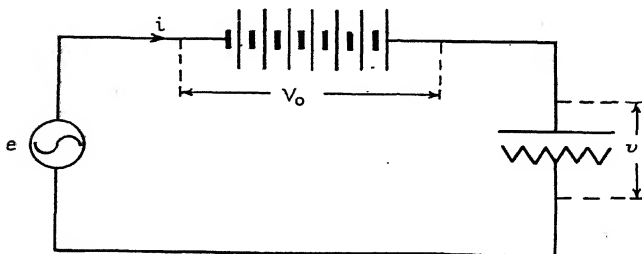
FIG. 9.03.

The particular problem to be discussed is indicated in Fig. 9.03. The tube is connected in series with a battery to a source of sinusoidal alternating current of negligible internal impedance. If $V_0 = 180$ volts and $e = 124.4 \sin \omega t$, what current flows in the circuit? Note first that current flows always because the battery voltage is greater than the peak voltage of the applied emf e. Thus the anode is always positive with respect to the cathode.

The following sections are devoted to several methods of calculating the current i.

9.03. Graphical Solution.—The graphical determination of i as a function of t is simple, but the information derived directly from it has limited value. The graphical solution is illustrated in Fig. 9.04. A vertical and a horizontal line are drawn through the point $v = 180$ volts, $i = 0.467$ ampere of the curve shown in
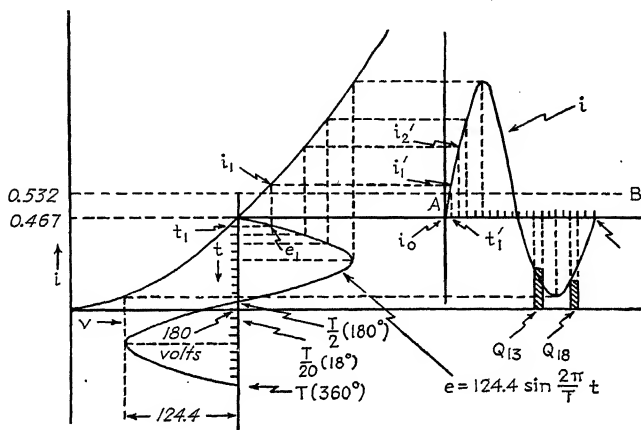
Fɪɢ. 9.04.

Fig. 9.02. A sine wave $e = 124.4 \sin \omega t$ is constructed on the vertical line, using any convenient length for the period T and the point $v = 180$ volts, $i = 0.467$ ampere for $t = 0$. A length corresponding to T is marked on the horizontal line. Both lengths corresponding to T are divided into n parts, n being an integer chosen for convenience (depending upon the divisions of the cross-section paper that is used) and for accuracy. In this case suppose that n is 20. Then one division corresponds to one-twentieth of one period, or 18 degrees if the period is 360 degrees. The function i is derived in the following manner. When $e = 0$ the voltage v across the tube is $V_0 = 180$ volts; the current i is therefore 0.467 amp. Mark the point designated i_0. The magnitude of the applied voltage at the instant $t_1 = T/20$ is e_1, i.e., it is the intersection of the horizontal line through t_1 and the curve e. The voltage across the tube at this instant is $e_1 + V_0$, and the current corresponding to this voltage

is i_1, *i.e.*, the point on the characteristic curve vertically above the point marked e_1. Next draw a horizontal line through i_1 and a vertical line through t_1', the point on the horizontal axis corresponding to the instant $T/20$. Their intersection is i_1', a point on the curve of the current. The process is repeated for the remaining instants $2T/20$, $3T/20$, . . . , $20T/20$. Finally a smooth curve is drawn through the points i_0, i_1', i_2', . . . , i_{20}'. *This curve i is the current through the circuit shown in Fig.* 9.03 *for one cycle.*

The current in the circuit is not an alternating current. In fact it is always positive; it is one form of *pulsating current.* With respect to the line $I_0 = 0.467$ ampere the wave is not representative of an alternating quantity because its average value, taking the line to be representative of "zero" current, is not zero. If a horizontal line AB is drawn through the ordinate $i = 0.532$ ampere, the wave *is* alternating with respect to this zero. However, the curve i is not a sine wave with respect to this ($i = 0.532$ ampere) axis. Therefore, the current cannot be represented by a time function comprising a constant added to a term for a sinusoidal quantity of a particular frequency.

The instantaneous value of the current can be determined from the graphical solution, but the current as a function of time cannot be specified directly in mathematical form from the graphical solution. The derivation of a useful mathematical expression of the current as a function of time from the graphical solution is described in Sec. 9.05, following an introductory discussion in Sec. 9.04.

9.04. Introductory Note on Fourier Series.—The analysis of electric circuits presented in this book shows that the electric current $i = f(t)$ generally satisfies the following conditions:

1. The current i at an instant t has precisely one value.
2. The current i never becomes infinite.
3. The rate of change of current with respect to time di/dt is zero only a finite number of times in a finite interval of time.
4. The rate of change of current with respect to time is never infinite.

The mathematician, if he were asked his opinion, would imply that i is always a well-behaved function of time, but he would not use the term "well-behaved" and he would rightly insist upon presenting several definitions and several rigorous mathe-

matical proofs of certain properties of $i = f(t)$. Leaving these matters to be studied[8] at the student's discretion, one important conclusion will be taken from the mathematical analyses and used for the solution of the problem proposed in Sec. 9.02.

This conclusion is that the current i can be expressed analytically in a time interval $0 \leq t \leq T$ by means of an infinite Fourier series:

$$(9.01) \quad i = f(t) = \sum_{n=0}^{n=\infty} A_n \sin \frac{2\pi n}{T} t$$

$$+ \sum_{n=0}^{n=\infty} B_n \cos \frac{2\pi n}{T} t \quad 0 \leq t \leq T$$

The implication is that the current, varying in any way it can vary in accordance with the conditions listed above during an interval of T seconds, may consist of a *direct-current component* B_0 ($n = 0$), a *fundamental component*

$$A_1 \sin \frac{2\pi}{T} t + B_1 \cos \frac{2\pi}{T} t \quad (n = 1),$$

and *harmonic components* $A_n \sin \frac{2\pi n}{T} t + B_n \cos \frac{2\pi n}{T} t \quad (n > 1)$

having frequencies equal to integral multiples of the fundamental frequency $1/T$. The Fourier series for the current i is practically useful only if (1) the coefficients A_n and B_n can be determined and (2) the coefficients A_n and B_n have values negligibly small from the practical point of view for all n greater than that n (for example $n = 7$) for which the calculator's patience decreases to a negligibly small value. In other words the calculation of the coefficients A_n and B_n is tedious if the number of coefficients that must be calculated is more than about 7.

An important property of the Fourier series representing currents that flow in electric circuits is that such series are *absolutely convergent*,[9] *i.e.*, the sum of all terms of both series for n greater than a particular integer n_1 is less than any assigned number δ no matter how small, for all values of t in the range $0 \leq t \leq T$. When the series are absolutely convergent, the terms may be rearranged in any convenient manner that will permit the calcu-

lation of A_n and B_n. The standard procedure for calculating these coefficients is described below.

Equation (9.01) can be rewritten as follows:

$$(9.02)\quad i = f(t) = A_1 \sin \frac{2\pi}{T} t + A_2 \sin \frac{4\pi}{T} t + A_3 \sin \frac{6\pi}{T} t$$

$$+ \cdots + B_0 + B_1 \cos \frac{2\pi}{T} t + B_2 \cos \frac{4\pi}{T} t + B_3 \cos \frac{6\pi}{T} t + \cdots$$

Multiply the equation, term by term, by $\sin \frac{2\pi m}{T} t$ (m is an integer) and integrate both sides of the equation from 0 to T. The result is

$$(9.03)\qquad \int_0^T f(t) \sin \frac{2\pi m}{T} t \, dt = \frac{T}{2} A_m \quad (m = 0, 1, 2, \cdots, n)$$

because

$$(9.04)\qquad \int_0^T \sin \frac{2\pi m}{T} t \cos \frac{2\pi n}{T} t \, dt = 0 \quad (m \text{ and } n \text{ are integers})$$

$$(9.05)\qquad \int_0^T \sin \frac{2\pi m}{T} t \sin \frac{2\pi n}{T} t \, dt = 0 \qquad \begin{cases} m \neq n \\ m = n = 0 \end{cases}$$

$$(9.06)\qquad \int_0^T \sin \frac{2\pi m}{T} t \sin \frac{2\pi n}{T} t \, dt = \frac{T}{2} \qquad m = n \neq 0$$

The student should prove the theorems Eqs. (9.04) to (9.06). To calculate A_m, the maximum value of the sine term of the mth harmonic, it is necessary to multiply the function $i = f(t)$ by $\sin \frac{2\pi m}{T} t$, integrate from 0 to T, and multiply the result by $\frac{2}{T}$.

The coefficients B_m are to be evaluated in a similar manner. Multiply Eq. (9.02), term by term, by $\cos \frac{2\pi m}{T} t$ and integrate both sides of the equation from 0 to T. The results are

$$(9.07)\qquad \int_0^T f(t) \cos \frac{2\pi m}{T} t \, dt = \frac{T}{2} B_m \qquad (m = 1, 2, \cdots, n)$$

and

$$(9.08)\qquad \int_0^T f(t) \, dt = T B_0 \qquad (m = 0)$$

because of the theorem of Eq. (9.03) above and

$$(9.09) \hspace{6cm} = n = 0$$

$$(9.10) \hspace{2.5cm} \int_0^T \hspace{1.5cm} \overline{T} \hspace{4cm} = n \neq 0$$

$$(9.11) \hspace{2.5cm} \int_0^T \cos \cdot \frac{}{T} t \cos \frac{2\pi n}{T} t \, dt = 0 \hspace{2.5cm} m \neq n$$

This method is applicable over the range 0 to T whether the function $f(t)$ is periodic or not. If the function is not periodic, the result of the Fourier analysis is valid only for the range $0 < t < T$. If the function is periodic with period T, the Fourier analysis is valid for all time.

When $i = f(t)$ is given in the form of a graph or an oscillogram, special means of applying the relations of Eqs. (9.03) to (9.08) to obtain the Fourier coefficients can be used. The methods are tedious and the accuracy of the results are vitiated by the difficulty of making accurate measurements of length in the graph or oscillogram. Nevertheless, these *harmonic analyses* are often made because no better method of solution is available.

9.05. Fourier Analysis of the Graph of a Periodic Current.—It is the purpose of this section to analyze the current wave form that was derived graphically in Sec. 9.03 (Fig. 9.04). Assuming that the wave form can be represented by a Fourier series (9.02), means must be found for evaluating the coefficients

$$(9.12) \hspace{3cm} = \frac{1}{T} \int_0^T f(t) \, dt$$

$$(9.13) \hspace{3cm} A_m = \frac{\llcorner}{T}$$

and

$$(9.14) \hspace{3cm} = \frac{2}{T} \int_0^T f(t) \cos$$

There are three common ways of evaluating these coefficients in terms of measurements of the graph of the current (Fig. 9.04):

1. The right-hand member of Eq. (9.12) can be obtained by measuring the area between the curve of i (Fig. 9.04) and the axis of abscissas $i = 0$ by means of a planimeter; this area, in (for example) square centimeters, divided by the product of the

length of the abscissa for one cycle and the length of the ordinate for one ampere is the *average current* for one cycle, *i.e.*, the number so calculated is B_0 in amperes.   Other curves can be constructed by multiplying each of several ordinates of   by $\sin \dfrac{2\pi m}{T} t$ or

cos   for $m = 1, 2, 3,$ The coefficients A_m and B_n

can be calculated in terms of these areas.

2. It can be shown that the integrals in the right-hand members of Eqs. (9.12) to (9.14) are related in a simple manner to the algebraic sum of certain ordinates of the curve $i = f(t)$.   Thus the coefficients A_m, B_m can be evaluated without measuring areas or plotting additional curves.   A method of this class is the *Fisher-Hinnen method.*[10]

3. An approximate *step-by-step integration,* such as that described below, may be used to evaluate the integrals of Eqs. (9.12) to (9.14).

Since the purpose of this chapter is to present from several points of view one simple non-linear circuit problem rather than the general theory of all such problems, only the step-by-step method of integration is presented below.

Step-by-step integration of the curve $i = f(t)$ of Fig. 9.04 is carried out in the following manner.   First divide the scale of abscissas from 0 to T into a convenient number (say 20) of equal parts, then list the ordinates of the curve i corresponding to these abscissas.   The second and fourth columns of the table below are respectively the twenty abscissas corresponding to a division of T into 20 parts and the ordinates of i for these 20 abscissas.   The third column is a list of the angles in degrees corresponding to the abscissas (360 degrees corresponds to T).   Note that the product of (for example) i_{13} and $\Delta t = T/20$ corresponds to the area Q_{13} in Fig. 9.04; similarly $i_{18} \Delta t$ corresponds to Q_{18}.   But Q_{13} is *larger* than the area between the curve i and the axis $i = 0$ in the interval $13T/20 \leq t \leq 14T/20$; Q_{18} is *smaller* than the corresponding area in the interval

$$\frac{18T}{20} \leq t \qquad \frac{19T}{20}.$$

Therefore,

(9.15) $i_{19} \, \Delta t$

Since $\Delta t = T/20$

$$(9.16) \qquad B_0 \approx \cdot \qquad\qquad + \qquad\qquad \text{amperes}$$

from Eq. (9.12). In this case the d-c component of i (B_0) is approximately 0.533 ampere. The accuracy of the result depends upon the accuracy of measurement of the ordinates of i and the number of divisions of T (in this case the number is 20).

DATA FOR HARMONIC ANALYSIS OF CURVE i, FIG. 9.04.

Ordinal number of abscissas	Abscissas (fraction of T)	Abscissas (θ deg.)	Ordinates (i amp.)	$\sin\theta$	$\cos\theta$	$\sin 2\theta$	$\cos 2\theta$
0	0	0	0.467	0	1.000	0	1.000
1	$T/20$	18	0.655	0.309	0.951	0.588	0.809
2	$2T/20$	36	0.833	0.588	0.809	0.951	0.309
3	$3T/20$	54	0.987	0.809	0.588	0.951	−0.309
4	$4T/20$	72	1.088	0.951	0.309	0.588	−0.809
5	$5T/20$	90	1.137	1.000	0	0	−1.000
6	$6T/20$	108	1.088	0.951	−0.309	−0.588	−0.809
7	$7T/20$	126	0.985	0.809	−0.588	−0.951	−0.309
8	$8T/20$	144	0.838	0.588	−0.809	−0.951	0.309
9	$9T/20$	162	0.654	0.309	−0.951	−0.588	0.809
10	$10T/20$	180	0.467	0	−1.000	0	1.000
11	$11T/20$	198	0.311	−0.309	−0.951	0.588	0.809
12	$12T/20$	216	0.193	−0.588	−0.809	0.951	0.309
13	$13T/20$	234	0.117	−0.809	−0.588	0.951	−0.309
14	$14T/20$	252	0.078	−0.951	−0.309	0.588	−0.809
15	$15T/20$	270	0.066	−1.000	0	0	−1.000
16	$16T/20$	288	0.079	−0.951	0.309	−0.588	−0.809
17	$17T/20$	306	0.118	−0.809	0.588	−0.951	−0.309
18	$18T/20$	324	0.194	−0.588	0.809	−0.951	0.309
19	$19T/20$	342	0.312	−0.309	0.951	−0.588	0.809

The coefficients A_m are expressed in Eq. (9.13) in terms of the integral of $f(t)\ \sin\dfrac{2\pi m}{T}t$ throughout the interval $0 \le t \le T$. For $m = 1$ the integral can be obtained approximately by listing (fifth column of the table) the values of $\sin\theta$ ($\theta =$ angle listed in the third column) and then calculating

(9.17) $i_0 \sin \theta_0 \, \Delta t + i_1 \sin \theta_1 \, \Delta t + \cdots i_{19} \sin \theta_{19} \, \Delta t$

$$= \int_0 f(t) \sin \frac{2\pi}{}$$

Therefore, from Eq. (9.13)

(9.18) $A_1 = \frac{1}{10}(i_0 \sin \theta_0 + i_1 \sin \theta_1 + \qquad \sin \theta_{19})$ amperes

In this case the fundamental component of i has a peak value (A_1) of 0.536 ampere. The coefficients A_2, A_3, A_4, . . . can be calculated in the same manner as A_1 if $\sin m\theta_0$, $\sin m\theta_1$, $\sin m\theta_2$ . . . , are substituted for $\sin \theta_0$, $\sin \theta_1$, $\sin \theta_2$, . . . , in Eq. (9.18). Values of $\sin 2\theta$ are listed in Column 7 of the table.

The coefficients of the cosine terms (B_m) are calculated in the manner described above for the coefficients A_m, except that $\cos m\theta$ is substituted in each case for $\sin m\theta$. Values of $\cos \theta$ and $\cos 2\theta$ are listed in Columns 6 and 8 of the table shown on page 223.

The values of the coefficients of the Fourier expansion of i, Fig. 9.04, calculated as described above, are

0.533 amp.	$B_1 \approx 0$
0.536 amp.	$B_2 \approx -0.064$ amp
0	$B_3 \approx 0$
0.005 amp.	

The relative accuracy of the value A_3 is good* with respect to the magnitude of B_0 or A_1, but it is poor with respect to the magnitude of A_3 itself. Thus errors inherent in the measurements of the ordinates of i and the choice of 20 divisions of T may cause errors of the order of ± 2 milliamperes in the results. Such an error is less than 0.4 per cent of B_0 or A_1 but it is 40 per cent of the calculated value of A_3. Furthermore, the third and higher harmonics are not likely to be practically significant because the sum of terms corresponding to A_1, B_0, and B_2 is nearly equal to the given curve i. The curve i is reproduced in Fig. 9.05; the sum of the terms involving A_1, B_0, and B_2 is marked, for each of several ordinates, by a small circle.

The Fourier analysis of the current for one cycle in the circuit shown in Fig. 9.03 is, therefore,

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* To replace these loose terms by a quantitative statement of relative and absolute errors would require a detailed study of the theory of error. See, for example, *Numerical Mathematical Analysis*; J. B. Scarborough, Chaps. I, XIV, XV, Johns Hopkins Press, Baltimore, 1930.

(9.19) $i = 0.533 + 0.536 \sin \dfrac{2\pi}{T} t - 0.064 \cos \dfrac{4\pi}{T} t + 0.005 \sin \dfrac{6\pi}{T} t$

Since the current is periodic with period T, the solution Eq. (9.19) holds for all time. The solution holds also for any frequency $f = 1/T$ for which Fig. 9.03 is representative of the electric circuit. For example, the solution is *not* valid for frequencies so high that the reactance of the condenser formed by the anode
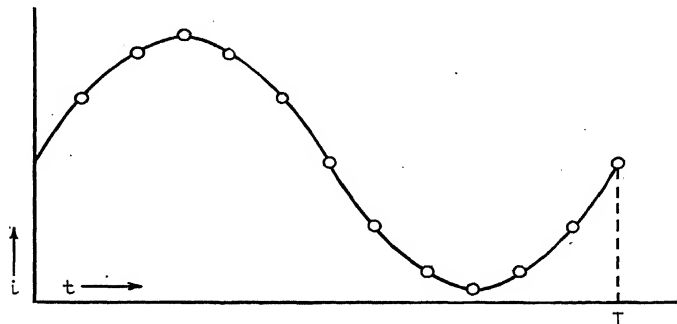
Fig. 9.05.

and cathode of the tube is of the same order of magnitude as the "resistance" of the tube.

9.06. Introductory Note on Taylor's Series.—A function $f(x)$ can be expanded in an absolutely convergent *Taylor's series* valid in an interval (of x) for which $f(x)$ and its derivatives $f'(x)$, $f''(x)$, . . . , $f^n(x)$ exist and are continuous.[11] This series is

(9.20) $f(x) = f(x_1) + \dfrac{f'(x_1)}{1!} (x - x_1) + \dfrac{f''(x_1)}{2!} (x - x_1)^2$
$$+ \dfrac{f'''(x_1)}{3!} (x - x_1)^3 + \cdots$$

in which x_1 is within the interval described above. Such an expansion is often useful in the analysis of electric-circuit problems.[12] The use of such a series for the solution of the problem proposed in Sec. 9.02 is discussed below.

9.07. Taylor's Series for the Current in Circuit of Fig. 9.03.— If the data of the static characteristic shown in Fig. 9.02 are plotted on log-log cross-section paper, the points fall on a straight

line from $v = 50$ volts to $v = 310$ volts. Thus within this interval

(9.21) $\log i = \alpha \log v + \beta$ $(50 \leq v \leq 310)$

(9.22) $i = K v^{\alpha}$ amperes

The two constants K and α can be determined by substituting data from Fig. 9.02. The current i as a function of v is

(9.23) $i = 7.63 \times 10^{-5} v^{1.68}$ amperes $(50 < v < 310)$

This function is continuous and its derivatives exist and are continuous throughout the intervals $50 < v < 310$. Therefore it can be expanded in a Taylor's series about a point, such as $v = 180$ volts, $i = 0.467$ ampere in the interval. The series is, from Eq. (9.20)

(9.24) $i = f(v) = f(V_0) + \dfrac{f'(V_0)}{1!}(v - V_0) + \dfrac{f''(V_0)}{2!}(v - V_0)^2$

$$\frac{f'''(V_0)}{3!}(v - V_0)^3 + \cdots$$

The function and its derivatives at $V_0 = 180$ volts can be evaluated from Eq. (9.23)

$$f(V_0) = 0.467$$
$$f'(V_0) = 4.35 \times 10^{-3}$$
$$\frac{f''(V_0)}{2!} = 0.83 \times 10^{-5}$$
$$\frac{f'''(V_0)}{3!} = -0.49 \times 10^{-8}$$

$$\frac{}{4!} \quad : 10^{-11}$$

Note also, from the circuit diagram, Fig. 9.03, that

(9.25) $e = v - V_0 = 124.4 \sin \dfrac{2\pi}{T} t$

Thus Eq. (9.24) is transformed by substitution of these quantities into a series that is a function of t

(9.26) $i = f(t) = 0.467 + 4.35 \times 10^{-3}\left(124.4 \sin \dfrac{2\pi}{T} t\right)$

$$+ 0.83 \times 10^{-5}\left(124.4 \sin \frac{2\pi}{T} t\right)^2 - 0.49 \times 10^{-8}\left(124.4 \sin \frac{2\pi}{T} t\right)^3$$

$$+ 10^{-11}\left(124.4 \sin \frac{2\pi}{T} t\right)^4 + \cdots$$

Each of the quantities involving a power of $\sin \frac{2\pi}{T} t$ can be expanded from the trigonometric identities

$$\sin^2 x = \tfrac{1}{2} - \tfrac{1}{2} \sin 2x$$
$$\sin^3 x = \tfrac{3}{4} \sin x - \tfrac{1}{4} \sin 3x$$
$$\sin^4 x = \tfrac{3}{8} - \tfrac{1}{2} \cos 2x + \tfrac{1}{8} \cos 4x$$

The final result is

$$(9.27) \quad i = 0.532 + 0.534 \sin \frac{2\pi}{T} t - 0.065 \cos \frac{4\pi}{T} t$$
$$+ 0.0024 \sin \frac{6\pi}{T} t + 0.00027 \cos \frac{8\pi}{T} t + \cdots$$

which is in close accord with Eq. (9.19) derived by Fourier analysis. The magnitudes of the third and fourth harmonics are doubtful, since relatively small deviations of i from the value of $7.63 \times 10^{-5} v^{1.68}$ would cause relatively large deviations in the coefficients for these harmonics.

9.08. Conclusion.—Although the illustration used throughout this chapter is a current consisting chiefly of a d-c component, a fundamental, and a second harmonic, the methods are applicable to currents represented by longer series. The example was chosen for simplicity, not for generality. One general result should, however, be noted. If a current consists of a fundamental and a series of harmonics of this fundamental, the rms value of the current is the square root of the sum of the squares of the rms values of the several terms; *i.e.*, if

$$i = \sqrt{2}\, I_1 \sin (\omega t + \quad + \sqrt{2}\, I_2 \sin 2(\omega t + \phi_2)$$
$$+ \sqrt{2}\, I_3 \sin 3(\omega t + \phi_3)$$

then

$$(9.28) \qquad I = (I_1^2 + I_2^2 + I_3^2 + \cdots)^{\frac{1}{2}}$$

The proof is left to the student. It should be noted also that the pairs of terms $A_m \sin \frac{2\pi m}{T} t$, $B_m \cos \frac{2\pi m}{T} t$ can be combined into a single sine or cosine term:

$$(9.29) \quad A_m \sin \frac{2\pi m}{T} t + B_m \cos \quad = C_m \sin \left(\frac{2\pi m}{T} t + \phi_m \right)$$
$$= C_m \cos$$

in which

$$C_m = (A_m^2 + B_m^2)^{\frac{1}{2}}; \qquad \phi_m = \tan^{-1} \frac{B_m}{A_m}; \qquad \phi_m' = \tan^{-1} \frac{A_m}{B_m}$$

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Problems

9.01. Assume the following conditions for a thin filament placed in an evacuated bulb: (1) the resistance of the filament at $T°C.$ is

$$R = R_{18}[1 + \alpha(T - 18)] \text{ ohms,}$$

R_{18} being the resistance at 18°C.; (2) the power input to the filament is directly proportional to $(T - 18)$.

a. Derive a formula for the current through the filament as a function of the applied voltage.

b. A carbon filament ($\alpha = -0.0005$ ohm/ohm°C.) normally operates at 2,000°C., with an applied voltage of 110 volts and power input of 50 watts. Plot a curve showing the relation between current and applied voltage from 0 volts to 125 volts.

c. Repeat *b* for a tungsten filament: $\alpha = +0.0045$ ohm/ohm°C.; $T = 2,600°C.$ for input power of 500 watts at 110 volts.

d. Criticize the assumptions upon which these solutions are based.

9.02. Thirty disks of thyrite, a ceramic semiconductor, are connected in series across the terminals of a 2,300-volt, 60-cycle source. The current in

one disk varies with the applied voltage thus

$$i = 2.13 \times 10^-$$ amperes

a. Plot a curve of current in the stack of 30 disks *vs.* time for one cycle of the 60-cycle wave.

b. If a resistance of 10,000 ohms is connected in series with the stack of 30 disks, how does the current through the combination vary with time?

c. What is the average power dissipation in the thyrite for *a* and *b* and in the 10,000-ohm resistance in *b*?

9.03. An iron filament in an evacuated glass bulb operates at a temperature of 500°C. when the applied voltage is 50 volts and the current through the filament is 1.7 amp. The device is connected as shown below.
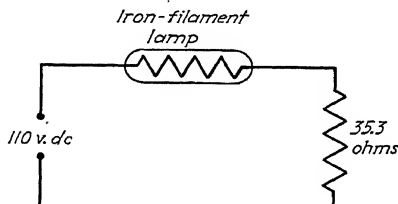
PROB. 9.03.

Calculate the voltage across the 35.3-ohm resistor for input voltages of 90, 95, 100, 105, 110, 115, 120, 125, 130 volts, assuming that the temperature coefficient of resistance of iron at 500°C. is 0.0147, and that the ambient temperature is 66°C.

9.04. The tube discussed in this chapter (IX) is connected as shown below.
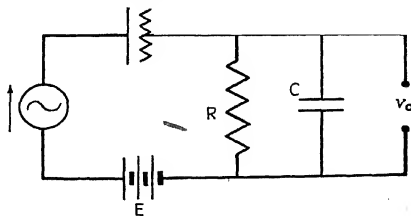
PROB. 9.04.

E	180 volts
R	10 ohms
C	

Calculate the output voltage v_o for an input voltage:

$$e = 124.4[\cos (10^6\pi t) + 0.5 \cos (10^6\pi - 10^3\pi)t + 0.5 \cos (10^6\pi$$

PART 2

Electrostatic and Electromagnetic Fields

CHAPTER X

THE ELECTROSTATIC FIELD

10.01. Introduction.—It is the purpose of Part 2 of this book to present briefly some of the fundamental aspects of the theory of the electrostatic and electromagnetic fields.

The fundamental technical bases of electrical engineering are the observable phenomena that can be correlated in terms of the magnitudes, positions, and motions of electric charges. When these charges are confined to electric-circuit elements, such as resistors, inductors, and condensers, some of the observable phenomena can be satisfactorily correlated by means of electric-circuit theory, which has been the subject of the first part of this book.

The theory developed in Part 1 is clearly inadequate for the analysis of many of the phenomena associated with the mechanical interaction of electrically charged bodies; with the mechanical and electromagnetic interaction of magnets, or current-carrying conductors, and of magnets and current-carrying conductors; with the use of vacuum tubes and gas-filled discharge tubes; with the radiation of electromagnetic energy from electric circuits; and with the production of light, ultraviolet radiation, and x rays by means of electrical devices. Investigations of these problems require the development and use of theories of far wider scope than those presented in Part 1. In fact, as some of these general theories are developed in this and subsequent chapters, it will be seen that the formulas derived from an elementary point of view in Part 1 are special cases of more general mathematical relations derived in Part 2. For example, the equations of circuit theory

$$(10.001) \qquad e = Ri \qquad \text{volts}$$

$$(10.002) \qquad q = Cv \qquad \text{coulombs}$$

$$(10.003) \qquad e = -L\frac{di}{dt} \qquad \text{volts}$$

are derivable from the theories of electric and magnetic fields.

The order in which the material of Part 2 is presented in this and the next four chapters is partly, but not completely, arbitrary. When electric charges are at rest relative to an observer, the pertinent phenomena and the theory that correlates them are relatively simple. Thus the elementary theory of *electrostatics* is the first subject of discussion in Part 2. The further development of this theory and its applications to a number of important engineering problems are the subjects of Chap. XI. When electric charges move with constant velocities relative to an observer, the phenomena and the theories that correlate them are more complicated than those of electrostatics. The phenomena and theories associated with charges that are accelerating relative to the observer are still more complex. These subjects are discussed in Chaps. XII, XIII, and XIV. The applications of the theories developed in Part 2 to several important engineering problems are discussed in Chaps. XV and XVI.

Modern atomic theory postulates two stable subatomic entities having electric charge, the electron (-1.59×10^{-19} coulomb) and the proton ($+1.59 \times 10^{-19}$ coulomb), as the building blocks of most electric and magnetic effects. The qualitative theory of electric-conduction currents, presented in Chap. III, is based on the assumption that some of the electrons of a piece of metal, or other conducting solid, can be caused to drift through the body of the metal and that such drifting of electrons constitutes an electric current. When some of the free electrons of a piece of metal have been removed from it, and no current flows in it, it is said to possess a *positive electrostatic charge.* The magnitude of this charge is the product of the number of free electrons removed from the piece of metal and the charge 1.59×10^{-19} coulomb. This follows from the inference—justified by experiment—that there were equal numbers of electrons and protons in the uncharged piece of metal before some of the free electrons were removed. When electrons are supplied to a neutral body, the body acquires a *negative electro-*

static charge. Charged bodies are subjected to forces of attraction or repulsion when they are in the neighborhood of other charged bodies; these are the subjects of electrostatics.

10.02. The Use of Vector Analysis to Represent Electrostatic and Electromagnetic Fields.—Electrostatic forces are exerted on electric charges placed near charged bodies. Such forces vary from point to point in space. It is therefore useful to postulate a set of axes so that points in space and the physical quantities such as force can be specified in terms of the coordinates represented by these axes. The concise methods of *vector analysis* are useful for this purpose.
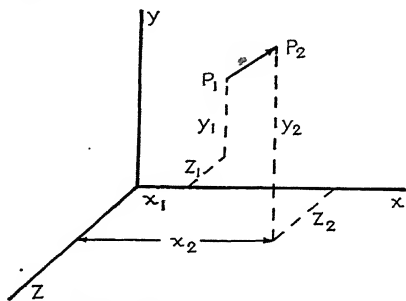
Fig. 10.01.—Right-hand cartesian coordinate system.

Some physical quantities, such as the volume-charge density (Sec. 10.04), can be specified, at each point (or, more precisely, within each small volume) in space, by a single number—the magnitude of the charge density $\bar{\rho}$ in coulombs per cubic meter or in any other agreed-upon units; such quantities are called *scalar quantities*. Other physical quantities, such as velocity, require for complete specification not only magnitude but also direction. Such quantities are called *vector quantities*. Vector quantities can be usefully represented by *vectors*,* *i.e.*, by agreed-

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* The concept of a vector as a mathematical representation of a physical quantity having magnitude and direction is a useful concept for the analysis of the material presented in this chapter and in the chapters that follow. More general definitions[2] are useful for the analysis of other problems in physics or engineering. Such general definitions have the added advantage —for the study of advanced problems—that they can be applied to "vectors" that are specified in terms of as many coordinates as may be desired, and not in terms of three coordinates, x, y, z, as in the examples given below and in the chapters following.

upon mathematical symbols, upon which various defined mathematical operations may be performed. A directed segment of a straight line is an example of a vector. Thus, in Fig. 10.01, the line P_1P_2, *in the direction $P_1 \rightarrow P_2$*, can be represented by the vector **A**, whose symbol **A** is printed in boldface type to distinguish it from the symbols of other quantities. *The magnitude* of the vector **A** is, in this case, the scalar quantity.

$$(10.004) \quad A =$$

and the direction of the vector **A** can be specified in terms of the cosines of the angles θ_x, θ_y, θ_z between the line P_1P_2 and lines through P_1 parallel respectively to the x, y, and z axes:

$$\cos \theta_x = \quad A$$

$$(10.005) \quad \cos \theta_y$$

$$\cos \theta_z =$$

The x component of a vector **A** is the scalar magnitude A_x of the projection of the line representing the vector into that line through one end of the vector line that is parallel to the x axis; corresponding definitions specify the y component A_y and the z component A_z of the vector **A**. In the simple example given above, the components of **A** are

$$(10.006) \quad \begin{array}{l} A_x = A \cos \theta_x = x_2 \\ A_y = A \cos \theta_y = y_2 \\ A_z = A \cos \theta_z = z_2 \end{array}$$

It follows from Eq. (10.004) that the magnitude A of a vector **A** is the square root of the sum of the squares of the components of the vector:

$$(10.007) \quad A = \quad \overline{A_z^2}$$

A second method for describing a vector by means of symbols makes use of *unit vectors*. These unit vectors specify *direction;* each of them has magnitude 1 and no dimensions. Three are required in a three-dimensional system; in a cartesian system they are, according to well-established convention

(10.008)
> **i**   unit vector in the $+x$ direction
> **j**   unit vector in the $+y$ direction
> **k**   unit vector in the $+z$ direction

Using these designations, the vector **A** can be specified by

$$(10.009) \qquad \mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z$$

Those concepts of vector analysis that are relevant to the development of a concise theory of electrostatic and electromagnetic fields are introduced in this and the following chapters with emphasis upon their significance from the physical point of view. A summary of this extremely important branch of mathematics is given in Appendix B (page 441).

10.03. Coulomb's Law.—Historically, the formulation of a quantitative theory of electrostatics began with measurements by Coulomb and others of the forces of attraction or repulsion between electrically charged bodies. These measurements are discussed in elementary courses in physics.

In the sections that follow, the theory of electrostatics is derived from the concepts of electric-charge density, electrostatic potential, and electrostatic-field intensity. This development *leads to* Coulomb's law of the forces between charged bodies as a logical consequence. An alternate procedure, not presented in this book, *starts with* Coulomb's law and derives mathematical relations among charge density, potential, and field intensity from the law of forces between charged bodies.

This section is devoted to a brief discussion of Coulomb's law. Its purposes are (1) to remind the student of the experimental basis of electrostatics; and (2) to show what form the Coulomb's law takes in the mks system of units.

Coulomb's law is a statement of the fundamental experimental facts upon which the theory of electrostatics is built, yet its mathematical expression contains a warning that its valid applications are not universal. Coulomb's law states that each of the forces of attraction or repulsion between two charged bodies immersed in a homogeneous medium of infinite extent—the bodies being small compared to their distance of separation—is directly proportional to their charges and inversely proportional to the square of their distance of separation, in the absence of changing magnetic fields.

From the macroscopic point of view the law is extremely useful and its applications are relatively simple. It is from this point of view that the law is considered in this section.

The forces between subatomic entities such as protons and electrons "separated" by distances of the order of 10^{-10} meter or less are difficult to subject to experimental measurement—direct or indirect. Thus the validity of Coulomb's law of force in these cases is not a subject to be settled by elementary analyses.

The inverse-square law for electric charges can be expressed as a vector relation

$$(10.010) \qquad\qquad \mathbf{F}_{12} = A \quad {}^{Q_1 Q_2}$$

in which $\mathbf{F}_{12}$ represents the force exerted *by* Q_1 on Q_2

   A is a constant depending on the medium in which Q_1 and Q_2 are placed, and upon the system of units
   Q_1, Q_2 are the charges under consideration
   r is the distance of separation of Q_1 and Q_2
   $\mathbf{r}_{12}$ is a unit vector (dimensionless) in the direction $Q_1 \rightarrow Q_2$

Thus if Q_1 and Q_2 are charges of the same sign ($+$ or $-$), Q_1 repels Q_2; if Q_1 and Q_2 are charges of the opposite sign ($+$ and $-$), Q_1 attracts Q_2. The double subscripts are commonly omitted from $\mathbf{F}$ and the symbol for the unit vector $\mathbf{r}_1$ for convenience. The forces $\mathbf{F}_{21}$, exerted by Q_2 on Q_1, is equal in magnitude and opposite in direction to the force $\mathbf{F}_{12}$.

If the mechanical units of force and distance have been specified, Eq. (10.010) may be used to define a unit of electric charge by assuming a value of A for one medium. Thus if the dyne is the unit of force, the centimeter is the unit of length, and A is assigned the value 1 for a vacuum, the unit charge is the *cgs electrostatic unit of charge*.

In the mks system the units of force (newton), length (meter), and charge (coulomb) are determined from other considerations (Chap. I). Thus the constant A is subject to direct or indirect experimental measurement. Furthermore, it is convenient for reasons that will appear later to put

$$(10.011) \qquad\qquad A =$$

for charges in a vacuum; ϵ_0 is called the *dielectric permittivity* of free space. The factor 4π is written explicitly in Eq. (10.011) in order that it shall not appear explicitly in the most used formulas of electrostatics; a system of units so formulated is called a *rationalized system.* This will suffice for the preliminary discussion of problems in which charges are distributed in a vacuum. Later it will be shown that problems involving distributions of charges in homogeneous insulating media other than free space can be solved by defining

$$(10.012) \qquad A = \frac{1}{4\pi\epsilon}$$

in which ϵ is the dielectric permittivity of the medium in question. The quantity ϵ/ϵ_0 is called the *dielectric constant* of the medium.

The force between two charges placed in an infinite homogeneous medium, subject to the qualifications and conventions discussed above, is

$$(10.013) \qquad \mathbf{F} =$$

and the magnitude of the force is

$$(10.014) \qquad F = \qquad \text{newton}$$

Note that, 4π being dimensionless, ϵ has dimensions.

$$\frac{Q^2}{Fr^2} \cdot \frac{(\text{coulomb})^2}{(\text{newton})(\text{meter})^2} \qquad \frac{(\text{coulomb})^2}{\text{joule}} \frac{1}{\text{meter}}$$

and that, from $W = \frac{1}{2}\frac{Q^2}{C}$, the dimensions of capacitance are

$$C \rightarrow \frac{Q^2}{W} \rightarrow \frac{(\text{coulomb})^2}{\text{joule}}$$

Therefore ϵ has the dimensions of *farads per meter.* The numerical value of ϵ can be measured. It is conventional, however, to specify (1) the numerical value of ϵ_0 for a vacuum, and (2) the numerical values of the dimensionless ratio ϵ/ϵ_0 for other substances. The dielectric permittivity of free space is

$$(10.015) \qquad \epsilon_0 = 8.85 \times 10^{-12} \qquad \text{farad per meter}$$

in the *rationalized* mks system of units.

The dielectric constants of most gases at normal temperature and pressure seldom exceed 1.001. Thus ϵ/ϵ_0 for air may be taken as unity without causing an error large enough to be objectionable in many practical cases. The dielectric constant of solids and liquids are discussed in Sec. 10.11.

Coulomb derived the ideal law Eq. (10.10) from measurements of the forces between charged bodies by means of a torsion balance. A few years before Coulomb's work was published, Cavendish predicted the inverse-square law on the basis of other kinds of experiments (see Sec. 10.04), but his analyses were not available until Maxwell published them about one hundred years later. Further experiments by Maxwell and recently by Plimpton and Lawton have shown that the exponent of r in Eq. (10.010), under certain conditions specified by them, does not differ from 2 by more than a few parts in a billion (10^9).

10.04. Electric-charge Density, Electrostatic Potential, and Electrostatic-field Intensity.—The theory of electrostatics is developed in this and the following sections on the basis of the assumption that electric charge is a continuous quantity. This procedure leads to a useful theory of many observed electrostatic phenomena, but other considerations may be required to apply this theory to problems involving distances corresponding to atomic dimensions.[1]

Designate by Δv_a a volume *each* of whose linear dimensions is smaller than the smallest positive number one may choose to select. To avoid circumlocution such a volume Δv_a will be referred to as an *infinitesimal of the third order*. Designate by ΔQ_a the excess of the total positive charge over the total negative charge in this volume. If the positive charge is actually in excess, then ΔQ_a is a positive number; if the negative charge is actually in excess, then ΔQ_a is a negative number. Designate by ρ_a the limit to which the ratio $\Delta Q_a/\Delta v_a$ tends as *each* of the linear dimensions of Δv_a tends to zero, *viz.*,

$$(10.016) \qquad \rho_a = \qquad\qquad \text{coulombs per cubic meter}$$

where η is the maximum linear dimension of Δv_a. The limit to which Δv_a tends as η tends to zero is a single point, say a point P_a. Therefore the ratio ρ defined by Eq. (10.016) is called *the volume density of the electric charge at the point P_a.*

In the ordinary notation of the calculus, Eq. (10.016) is written

$$(10.017) \qquad \rho_a = \frac{dQ_a}{dv_a}$$

and both dQ_a and dv_a are called *differentials*, or *infinitesimals*. From Eq. (10.017) it follows that if ρ_a has a value at any number of points P_a in a volume v_a, then the total net positive charge in this volume v_a is the sum of the products $\rho_a\, dv_a$ for each of these points. Such a sum, in the usual notation of the calculus, is written

$$(10.018) \qquad Q_a = \int \rho_a\, dv. \qquad\qquad \text{coulombs}$$

Consider now any point P within or outside the volume v_a and form the integral

$$(10.019) \qquad = \frac{1}{4\pi\epsilon_0} \int_{v_a}$$

where ρ_a is the charge density at P_a, dv_a is an infinitesimal volume which encloses P_a and each of whose linear dimensions is an infinitesimal of the first order, and r is the distance from P_a to P, and the integral sign indicates the sum of the products $\rho_a\, dv_a/r$ for all of the points P_a within the volume v_a at which ρ_a has a value different from zero, and ϵ_0 is a constant. When ϵ_0 is the numerical value (approximate)

$$(10.020) \qquad \epsilon_0 = 8.85 \times 10^{-12} \qquad\qquad \text{farad per meter}$$

and all linear dimensions are expressed in meters and the charge density ρ_a is expressed in coulombs per cubic meter, then the quantity V defined by Eq. (10.019) is called the *absolute electrostatic potential, in volts, at the point P, due to all the electric charges in the volume v_a.* The constant ϵ_0 is called the electric *permittivity of free space*, or more briefly, the *permittivity* of free space.

For a given distribution of electric charges in a finite volume v_a at no point of which ρ_a has an infinite value, the function V defined by Eq. (10.019) is a finite, single-valued, and continuous function of the three space coordinates of the point P. In particular, when the position of P is expressed by the three coordinates x, y, z in a rectangular cartesian coordinate system, and the position of any point P_a in the volume v_a is expressed by

the coordinates x_a, y_a, z_a in the same system then

$$(10.021) \qquad r = \overline{(z}$$

and V is then a function of the three variables (x, y, z). Note, however, that V is not a function of x_a, y_a, z_a, since these quantities disappear when the integration indicated has been performed. On the other hand, for a given distribution of charges in the volume v_a the parameters that appear in the function

$$(10.022) \qquad V = f(x, y, z) =$$

do depend upon the charge density at each point in the volume v_a and also upon the location of the points at which ρ_a has a value different from zero; *i.e.*, upon the *magnitudes and positions of the charges in the volume* v_a.

The usefulness of the function V defined by Eq. (10.019) lies in the fact that, except when the distribution of the electric charges in a specified region is varying rapidly with time, the drop of electric potential V_{12} (as defined in Sec. 1.06) from a specified point P_1 to a specified point P_2 is equal to the difference $(V_1 - V_2)$, where V_1 and V_2 are the values of the absolute electrostatic potential at these two points, *viz.*,

$$(10.023) \qquad V_{12} = V_1 - V_2$$

Moreover, under the same conditions one may express, in terms of this absolute potential, the mechanical forces exerted by one electric charge on another, provided the order of magnitude of the distance between these charges is sufficiently great compared with the linear dimensions of a molecule.

In the analysis of the mechanical forces between electric charges attributable solely to their magnitudes and position, it is convenient to make use of other functions readily derived from the absolute electrostatic potential. Note first of all that, by assigning dimensions to the constant ϵ_0 in Eq. (10.019), the physical quantity whose magnitude is represented by the symbol V in that equality may be made the same as those of *work per unit charge*. The partial derivative of V with respect to any one of the three coordinates x, y, or z then has the dimensions of *force per unit charge*. Designate the *negatives* of these partial

derivatives by the symbols E_x, E_y, E_z, *viz.*,

$$(10.024) \qquad E_y = -\frac{\partial V}{\partial y} \qquad \text{volts per meter}$$

The vector **E** whose components in the positive sense of the x axis, y axis, and z axis are equal respectively to the values of these three partial derivatives, at any point P, is defined as the *electrostatic-field intensity* at this point due to the charges that give rise to the electrostatic potential V at this point. Wherever there exists an electrostatic intensity different from zero there is said to exist an *electrostatic field*.

In the notation of vector analysis the electrostatic intensity just defined is written

$$(10.025) \qquad \mathbf{E} = - \qquad + \mathbf{j}\frac{\partial V}{\partial y} +: \frac{\partial V}{\partial z} \Big) \text{ volts per meter}$$

where the symbols **i**, **j**, **k** indicate that the numbers which they "multiply" are respectively the algebraic values of the components of **E** in the positive senses of the x axis, y axis, and z axis respectively, and the plus signs in the parenthesis indicate *vector* (not algebraic) addition. A still briefer notation to indicate the relation between the electrostatic intensity **E** and the electrostatic potential is

$$(10.026) \qquad \mathbf{E} = -\boldsymbol{\nabla} V \qquad \text{volts per meter}$$

where the symbol $\boldsymbol{\nabla}$, called *del*, stands for the *vector operator*

$$(10.027) \qquad \boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

The vector that has the same magnitude as **E** but the *opposite sense* is called the *gradient of the scalar function V* and is sometimes indicated by the notation **grad** V, *viz.*,

$$(10.028)$$

Relations among $\boldsymbol{\nabla} V$, **E**, and the components of $\boldsymbol{\nabla} V$ are shown in

Fig. 10.02. In this notation the electrostatic intensity at any
point P corresponding to an electrostatic potential V at this
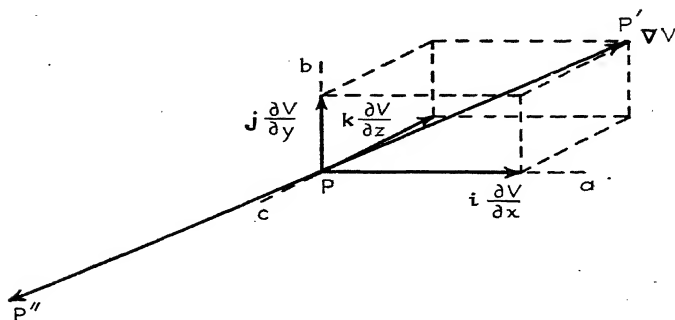
point is

$$(10.029) \qquad \mathbf{E} = -\ \mathbf{grad}\ V = -\boldsymbol{\nabla}V \qquad \text{volts per meter}$$

10.05. Electrostatic–field Intensity and Mechanical Force.—
Irrespective of the notation used to indicate the relation between
electrostatic intensity and electrostatic potential, the following
relations are in accord with the original experiments of Coulomb,
and subsequent experiments of many others, in which the
mechanical forces between charged bodies were actually meas-
ured. Designate by V the electrostatic potential at a point P
due to all the electric charges in space other than those in a body
of volume dv, which contains the point P and which has linear
dimensions of an order of magnitude many times the linear
dimensions of a molecule, but yet is sufficiently small to be
considered mathematically as being an *infinitesimal* volume
"located" at the point P. Then the mechanical force exerted
on this element of the body B due to all these other charges
has the magnitude

$$(10.030) \qquad dF = E\ dQ \qquad \text{newtons}$$

where E is the magnitude of the electric intensity $\mathbf{E}$, and the
direction of this force is the same as that of $\mathbf{E}$ when dQ is positive
and opposite to that of $\mathbf{E}$ when dQ is negative.

The component of the mechanical force dF on the elementary volume dv of the body B, in the positive sense of the x axis is

$$(10.031) \qquad dF_x = E_x\,dQ = -\frac{\partial V}{\partial x},$$

and similarly for the components in the positive senses of the y axis and z axis. Designate by v the volume of all those parts of the body B in which the excess of the positive over the negative charge, in any infinitesimal portion dv of this volume, is different from zero. Then the total component of the mechanical force on B in the positive sense of the x axis is the volume integral

$$(10.032) \qquad F_x = \int_v \rho E_x\,dv \qquad\qquad \text{newtons}$$

where ρ is the charge density in the infinitesimal volume dv and E_x is the component of the electrostatic intensity at the point P at which this infinitesimal volume is located, due to all the charges, *other than those in dv*, which are near enough to this point to affect appreciably the electrostatic intensity at this point. A similar relation holds for the components F_y and F_z of the resultant force on the body B. This resultant force therefore has the magnitude

$$(10.033) \qquad F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

and its direction is determined by the three direction cosines

$$(10.034) \qquad \begin{aligned} \cos\theta_x &= \frac{F_x}{F} \\ \cos\theta_y &= \frac{F_y}{F} \\ \cos\theta_z &= \frac{F_z}{F} \end{aligned}$$

in accordance with the vector concepts discussed in Sec. 10.02.

10.06. Electrostatic Energy.—Consider an infinitesimal charge dQ that occupies a volume (either in free space or in matter) each of whose linear dimensions is infinitesimal, and imagine this charge dQ to move from a point P_1 to a point P_2 at which the electrostatic potential due to all charges has respectively the values V_1 and V_2. Let V be the electrostatic potential, due to these other charges, at any point P on the path along which dQ

moves.   At the successive points P of such a path the electro-
static intensity is

(10.035) $\mathbf{E} = -\nabla V$ volts per meter

and the magnitude of the mechanical force exerted on the moving
charge dQ attributable to this electrostatic intensity $\mathbf{E}$ is $E\,dQ$.
The work done by this force on dQ corresponding to an infini-
tesimal displacement ds along the chosen path from P_1 to P_2
is $(E\,dQ)\cos\theta\,ds$, where θ is the angle between the direction
of $\mathbf{E}$ and the direction of $\mathbf{ds}$.   The direction cosines of ds are
respectively

(10.036)
$$\frac{dx}{ds}$$
$$\frac{dy}{ds}$$
$$\frac{dz}{ds}$$

where ds is the magnitude of $\mathbf{ds}$, *viz.*, the *positive* number

(10.037) $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$ meter

and the direction cosines of $\mathbf{E}$ are respectively

(10.038)
$$\frac{E_x}{E}$$
$$\frac{E_y}{E}$$
$$\frac{E_z}{E}$$

where E is the magnitude of $\mathbf{E}$, *viz.*, the positive number

(10.039) $E = \sqrt{E_x^2 + E_y^2 + E_z^2}$ volts per meter

Hence, from analytic geometry

(10.040) $\cos\theta = \dfrac{1}{E\,ds}(E_x\,dx + E_y\,dy + E_z\,dz)$

$$1 \quad \Big(\frac{\partial V}{} , \quad \frac{\partial V}{} , \quad \frac{\partial V}{}$$

The quantity in the parenthesis in the last member of Eq.
(10.040) is the total differential dV of the function V correspond-

ing to a change in the position of the volume dv from the point whose coordinates are (x, y, z) to the point whose coordinates are $(x + dx, \ y + dy, \ z + dz)$. Therefore Eq. (10.040) may be written

(10.041) $E \cos \theta \, ds = -dV$

Hence the work *per unit positive* charge done on an infinitesimal charge dQ by the force attributable to the electrostatic intensity at dQ, when dQ moves an infinitesimal distance ds in *any* direction in the electrostatic field, is the corresponding value of the negative of the total differential in this direction of the electrostatic potential V from which the electrostatic intensity **E** is derived.

Therefore, the total work done *per unit positive charge* when an infinitesimal charge dQ moves from a point P_1 to P_2 in this field, by the forces attributable to the electrostatic intensity at the successive points in its path of motion, is

(10.042)   volts

where V_1 and V_2 are respectively the values of the electrostatic potential at the beginning and end of the path of travel. If V_2 is less than V_1, then $(V_1 - V_2)$ is *positive,* and there is said to·be a *drop* of electrostatic potential in the sense of this motion. If V_2 is greater than V_1 then $(V_1 - V_2)$ is *negative,* and there is said to be a *rise* of electrostatic potential in the sense of this motion.

Since the function V defined by Eq. (10.019) is *single-valued* with respect to each of the coordinates x, y, z that define the position of the point at which the electrostatic potential has the value V, it follows that if the charge dQ moves around any *closed* path from a point P_1 back to this same point, then $(V_1 - V_2)$ is *zero,* and therefore *the work done per unit positive charge on this moving charge dQ, attributable solely to the electrostatic intensity E, is zero.*

10.07. The Scalar Product of Two Vectors and the Circulation of a Vector.—Consider any integral of the form

(10.043)

where ds is an infinitesimal segment of a closed loop C measured in a specified circulatory sense (say counterclockwise) around this

loop, and E is the magnitude of a vector at the point at which ds is located, and θ is the angle between the direction of this vector and the direction of ds. Such an integral is called the *circulation* of the vector **E** around this closed loop. The circulation of any vector **E** that is the gradient of an algebraic or scalar function V is always *zero*, provided V and its partial derivatives with respect to the three space coordinates x, y, z are finite and single-valued functions of these coordinates at each point of the loop C.

In vector analysis, a product of the form

(10.044) $AB \cos \theta$

where A and B are the magnitudes of two vectors **A** and **B,** and θ is the angle between them is usually written $\mathbf{A} \cdot \mathbf{B}$ and is called the *dot product* or scalar product of these two vectors. In terms of the components of these two vectors in the positive sense of the three coordinate axes,

(10.045) $\mathbf{A} \cdot \mathbf{B} = AB \cos \theta = A_x B_x + A_y B_y + A_z B_z$

When **A** and **B** are expressed in the notation

(10.046) $\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z$
(10.047) $\mathbf{B} = \qquad\qquad \mathbf{k}B_z$

the dot product is equivalent to the product of the two "polynomials" represented by the right-hand members of these two expressions, provided the products $\mathbf{i} \cdot \mathbf{i}$, $\mathbf{j} \cdot \mathbf{j}$, and $\mathbf{k} \cdot \mathbf{k}$ are each interpreted as unity, and provided the products $\mathbf{i} \cdot \mathbf{j}$, $\mathbf{j} \cdot \mathbf{k}$, and $\mathbf{k} \cdot \mathbf{i}$ are each interpreted as zero. These relations are equivalent to considering **i, j, k** as three mutually perpendicular *unit vectors* (*dimensionless*), according to Sec. 10.02.

Since the directed segment **ds** of any curve in three-dimensional space is the vector

(10.048) $\mathbf{ds} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz$

the circulation of any vector

(10.049) $\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z$

may be expressed as the integral of the dot product $\mathbf{A} \cdot \mathbf{ds}$ around the specified closed loop C, *viz.*,

(10.050) $\oint_C A \cos \theta \, ds = \int_C \mathbf{A} \cdot \mathbf{ds} = \oint_C (A_x \, dx + A_y \, dy + A_z \, dz)$

10.08. The Vector Operator ∇ and the Scalar Operator ∇^2.—
The concept of a dot product may also be extended to the vector
operator ∇. For example, if **A** is any vector expressed in the
form of Eq. (10.046) and ∇ is the operator defined by Eq.
(10.027), then in the notation just defined $\nabla \cdot \mathbf{A}$ signifies the
scalar quantity

$$(10.051) \qquad \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial z} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

This scalar quantity is called the *divergence* of the vector **A**.
Similarly the dot product of ∇ by itself, usually written ∇^2, is the
scalar operator

$$(10.052) \qquad \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

In the particular case of a vector **E** that is the negative of the
gradient of a scalar potential,

$$(10.053) \qquad \nabla \cdot \mathbf{E} = -\nabla \cdot \nabla V = -\nabla^2 V$$
where

(10.054)

It can be shown* (see Appendix B) that when V has the mathe-
matical form defined by Eq. (10.019) then

(10.055)

where ρ is the value of ρ_a at the particular point P_a in the volume
v_a that coincides with the point P at which V is given by Eq.
(10.019). Therefore, from Eq. (10.053), *the divergence of the
electrostatic intensity* **E** *at any point P in matter or in free space is
equal to the electric–charge density at this point divided by the
permittivity of free space, viz.*,

$$(10.056) \qquad \nabla \cdot \mathbf{E} = \frac{\rho}{\ }$$

Where there is *no electric–charge density, i.e.,* at every point at
which ρ is *zero,* the divergence of the electrostatic intensity **E** is

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* Provided ρ and its first- and second-order partial derivatives are each
finite and single-valued functions of x, y, and z.

zero, and therefore Eq. (10.054) becomes

$$(10.057) \qquad \overline{\partial x^2} \qquad \partial y^2 \qquad {}^{\shortmid} \quad \partial z^2 = 0$$

This form of partial differential equation is known as *Laplace's equation.*

Wherever the electric-charge density has a value ρ *different from zero,* then from Eqs. (10.054) and (10.055),

$$(10.058) \qquad\qquad \smile x \qquad \smile y \qquad \smile z \qquad \epsilon_0$$

This form of partial differential equation is known as *Poisson's equation.*

10.09. Flux of a Vector.—Another extremely useful concept in the analysis of electric and magnetic fields is that of the *flux of a vector.* Consider any surface dS whose linear dimensions are infinitesimal. Such a surface may always be considered as an infinitesimal area of a plane. Imagine a line perpendicular to this plane at any point of dS, and choose as the positive sense of this line the sense of a perpendicular drawn from dS to a point on a specified side of this plane. Such a line will be referred to as the *normal to dS in a chosen sense.* Designate by $\mathbf{dS}$ the vector whose magnitude is equal to dS and which has the same direction cosines $\cos \theta_x$, $\cos \theta_y$, $\cos \theta_z$ as this normal, *viz.,*

$$(10.059) \qquad \mathbf{dS} = (\mathbf{i} \cos \theta_x + \mathbf{j} \cos \theta_y + \mathbf{k} \cos \theta_z)\, dS$$

Let

$$(10.060) \qquad \mathbf{A} = \mathbf{i} A_x + \mathbf{j} A_y + \mathbf{k} A_z$$

be any vector that has a magnitude A and a specified direction at the infinitesimal surface dS, and let θ be the angle between the direction of this vector and the direction of the vector $\mathbf{dS}$. Then the product

$$(10.061) \qquad A \cos \theta\, dS = \mathbf{A} \cdot \mathbf{dS}$$

is called the *flux* of the vector $\mathbf{A}$ through the surface dS in the sense of the specified normal to dS.

Any surface S, no matter what its shape or dimensions are, may be considered as made up of contiguous differential areas dS, each of which in general lies in a different plane. Designate by S_1 and S_2 the two "sides" of this surface and designate by $\mathbf{dS}$ the vector that at any point of this surface has the magnitude dS and

the direction of the normal to dS in the sense from S_1 to S_2. Then the sum of the infinitesimal fluxes in the sense from S_1 to S_2 through all the infinitesimal areas that make up the surface S, *viz.*,

$$(10.062) \qquad \int_S \mathbf{A} \cdot d\mathbf{S} = \int_S A \cos \theta \, dS$$

is called the *total flux* of the vector $\mathbf{A}$ through the surface S in the sense from S_1 to S_2. The flux through S in the opposite sense, *viz.*, from S_2 to S_1, is the *negative* of the flux through S in the sense from S_1 to S_2. Note that the flux of a vector is a scalar quantity.

When the surface S is the closed surface or surfaces that bound a specified volume, and the normal at each point of this surface is considered as being drawn from a point just inside to a point just outside the enclosed volume, then the total flux defined by Eq. (10.062) is called the total *outward flux* of $\mathbf{A}$ from the specified volume. The total *inward flux* of $\mathbf{A}$ into a specified volume is the *negative* of the total outward flux of $\mathbf{A}$ from this volume.

By making use of the fact that the flux of a vector through a given surface in a specified sense is numerically equal to, but of opposite algebraic sign to, the flux of the same vector through the same surface in the opposite sense, it can be shown that the total outward flux of a vector $\mathbf{A}$ through the complete *bounding* surface S of a volume v is equal to the volume integral of the divergence $\nabla \cdot \mathbf{A}$ of this vector throughout this volume, *viz.*,

(10.063)

This relation is known as Gauss's theorem (see Appendix B for proof). In making use of this theorem one must always keep in mind that the surface indicated by the symbol S at the base of the integral sign in the left-hand member is the *entire bounding surface* of the volume indicated by the symbol v at the base of the integral in the right-hand member.

Since the divergence of the electrostatic intensity $\mathbf{E}$ at any point is equal to the electric-charge density at this point divided by the permittivity ϵ_0 of free space, the outward flux of $\mathbf{E}$ through the bounding surface of any volume v is

$$(10.064) \qquad \int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0}$$

where Q is the net electric charge within this volume. It can be

shown that this relation is valid even though, at certain points, the charge density is discontinuous.

If all the electric charge in a given volume v is concentrated in a very small volume at a point P_1 that is at a relatively very great distance r from any specified point P, then from symmetry one may assume that the vector $\mathbf{E}$ has a constant magnitude at every point on a sphere of radius r with P_1 as its center and is in the direction of the line from P_1 to P.   Under these conditions the left-hand member of Eq. (10.064) becomes $4\pi r^2 E$, and therefore

$$(10.065) \qquad\qquad E = \frac{1}{,} \frac{Q}{r^2} \qquad\qquad \text{volts per meter}$$

Since this relation is true no matter how small v is chosen, one frequently refers to this formula as the formula for the electrostatic intensity due to a *point charge Q, viz.,* to an electric charge of Q coulombs concentrated in a geometrical point.   Note that the direction of the electrostatic intensity $\mathbf{E}$ whose magnitude is given by Eq. (10.065) is the direction of the line drawn from the point charge Q to the point P when Q is positive and is the opposite direction when Q is negative.

The electrostatic potential V at the point P is a function whose negative gradient is the vector $\mathbf{E}$, the magnitude of which is given by Eq. (10.065).   The potential is

$$(10.066) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{volts}$$

10.10. Electric Charges in Free Space.—The importance of the scalar potential V for solving problems in electrostatics can be illustrated by the problem of calculating the electrostatic field $\mathbf{E}$ at a point P in a region where many charges $Q_1, Q_2, \ldots, Q_n$ are placed at points $(x_1 y_1 z_1), (x_2 y_2 z_2), \ldots, (x_n y_n z_n)$ in a system of rectangular coordinates x, y, z.   This problem is illustrated in Fig. 10.03.

If the contribution to the field $\mathbf{E}$ at P by the charge Q_1 is that calculated from Eq. (10.065) irrespective of the presence or absence of charges $Q_2, Q_3, \ldots, Q_n$, then the n contributions to $\mathbf{E}$ by the n charges can be calculated separately and added vectorially.   Experiment justifies the assumption that this supposition is true.   Thus there is a principle of superposition in electrostatics as well as in circuit theory.   Differential equations

relating V, $\mathbf{E}$, and the charge density ρ are, therefore, linear differential equations [see Eqs. (10.055) and (10.056)].

To set up the n contributions to the field $\mathbf{E}$ at the point P of Fig. 10.03, produced by the n charges Q_1, Q_2, Q_3, . . . , Q_n, and *then to add them vectorially*, is clearly a laborious procedure.   On the other hand, the algebraic sum of the contributions to the
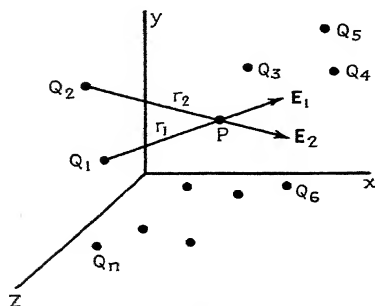
Fig. 10.03.

potential V at the point P is not hard to evaluate.   If $\mathbf{E}$ can be determined from V, the potential V will have served the excellent purpose of eliminating the need for a complicated vectorial addition for the determination of $\mathbf{E}$.

In terms of these concepts, the calculation of $\mathbf{E}$ at *any* point P, Fig. 10.03 proceeds as follows: (1) calculate the contribution V_1 to the potential V at a point

P whose coordinates are x, y, z of the charge Q_1 by expressing Eq. (10.066) in terms of the coordinates x_1, y_1, z_1 of the charge Q_1

$$(10.067) \quad V_1 = \frac{Q_1}{4\pi\epsilon_0 \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}}$$

and similarly for the other charges Q_2, Q_3, . . . , Q_n, which produce contributions V_2, V_3, . . . , V_n to the potential V at point P; (2) calculate V by adding algebraically V_1, V_2, . . . , V_n; (3) calculate the components of $\mathbf{E}$ at P by taking the negative partial derivatives in accordance with Eq. (10.024); (4) calculate the magnitude and direction of $\mathbf{E}$ from Eqs. (10.007) and (10.006). Note that the result is the value of $\mathbf{E}$ at any point $P(x, y, z)$.   To find the numerical value of $\mathbf{E}$ at a particular point having coordinates x_0, y_0, z_0, the numerical values of these coordinates can be substituted for x, y, z.

As a second example, suppose that a hollow (closed-surface) conductor is given a charge Q coulombs and that there is no charge within the hollow space.   How does the charge distribute itself in the conductor?   Soon after the charge is supplied to the conductor, the charge "distributes itself" and then maintains this distribution; *i.e.*, it is a problem in electrostatics.

If there were a difference of potential between any two points of the conductor, a current would flow. Since no such current flows, the conductor material is all at the same potential, V. Then the inner surface of the conductor is an equipotential surface. Assume that a closed equipotential surface wholly within the cavity has a potential V_1. If V_1 is greater or less than V, it follows from Gauss's law, Eq. (10.064), that there is a charge within the surface. This is contrary to the stipulated conditions of the problem. Therefore, all points within the cavity have the same potential V as the conductor *and there is no electric field within the cavity.* If there were charge on the inner wall of the cavity, there would be an electric field either in the cavity or in the conductor; both are contrary to previously stated assumptions or conclusions. Therefore the charge is on the outer "surface" of the hollow conductor. It is convenient, therefore, to specify in some problems the *surface charge density* σ at a point as the ratio of the charge in a small volume surrounding the point to the volume, as the latter decreases, assuming that the thickness of this volume is always very small. This concept is developed rigorously in Sec. 10.11, from a point of view useful for solving many problems in electrostatics that do not involve volumes having dimensions as small as atomic dimensions. From the point of view of atomic physics, the surface charge is discussed in one of the references.[5]

Note that if there were a tangential component of the electric-field intensity at the surface of a conductor, a current would flow in the conductor. Since current does not flow, *the electric-field intensity* $\mathbf{E}$ *is always perpendicular to the charged surface of a conductor in an electrostatic field.*

If the force between two charges were not inversely proportional to the square of the distance of separation, the electric-field intensity inside a (hollow closed-surface) conductor would not be zero.[6] Thus Cavendish, Maxwell, and Plimpton and Lawton[4] demonstrated the inverse-square law by attempting to measure the field within a charged hollow sphere by successively more sensitive means.* The experiments of Plimpton and Lawton show that the exponent of r in Eq. (10.013) or (10.065)

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* Note that this process consists of trying to measure something that is thought to be zero in such a manner that more and more significant figures are added to the number 0.0000 . . . .

does not differ by more than a few parts in one billion from 2, as noted above.

The next problem concerns an initially uncharged spherical conductor of radius R that is to be placed in a region where the field is originally uniform. This problem is here introduced, not that it has any particular practical application, but to illustrate the usefulness of Laplace's equation and the boundary conditions in the solution of electrostatic problems. When the sphere is introduced into the field, electrons move in the conductor and
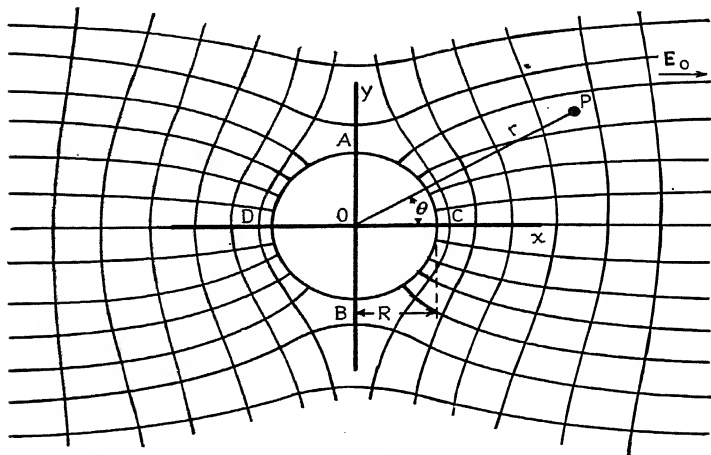
FIG. 10.04.

then come to rest. How are they redistributed? Figure 10.04 is a diagram of the problem to be solved. Note that (1) there can be a net charge only on the surface of the spherical conductor, since otherwise there would be a field and therefore a current in the conductor; (2) the electrostatic-field intensity **E** is perpendicular to the surface of the sphere (*i.e.*, radial), or zero at every point on the sphere; and (3) the problem has cylindrical symmetry about the x axis so that only one plane (for example, the plane of the paper in Fig. 10.04) need be considered. The last deduction, (3), reduces the problem to one of two dimensions (x, y). Furthermore, it is convenient in this case to use polar coordinates (r, θ) rather than cartesian coordinates (x, y).

The problem, therefore, is to find a solution $V(r, \theta)$ to Laplace's equation in the region surrounding the sphere, to derive the vector field of $\mathbf{E}$ from the scalar V, and to calculate $\sigma(\theta)$ the surface charge density on the surface of the sphere.

Note that Laplace's equation for three dimensions in polar coordinates, when there is no variation of V with the angle ϕ measured from Oy into the paper, Fig. 10.04, is

$$(10.068)^* \qquad \frac{\partial V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial V}{\partial \theta^2} + \frac{\cot \theta}{r^2}\frac{\partial V}{\partial \theta} = 0$$

In Eq. (10.068), V is the potential at any point outside of the sphere. Since V is a continuous function, it must approach the potential V_0 (the assumed potential of the sphere) at the surface of the sphere, and the gradient of V becomes $-\mathbf{E}_0$ at all points far from the sphere. Thus the boundary conditions are

$$(10.069) \qquad\qquad V = V_0 \qquad \text{for} \qquad r = R$$
$$(10.070) \qquad V = -E_0 r \cos \theta \qquad \text{for} \qquad r >> R$$

Equation (10.070) represents the condition that the value of $\mathbf{E}$, at distances very large compared to the radius R of the sphere, is $\mathbf{E}_0$, the original field intensity; $\mathbf{E}_0$ is parallel to Ox in Fig. 10.04. To prove that Eq. (10.070) represents the boundary condition, note that $-\boldsymbol{\nabla} V$ in polar coordinates r, θ, ϕ, *when* $\partial V/\partial \phi = 0$ is, from Appendix C,

$$(10.071)$$

Substituting Eq. (10.070) in Eq. (10.071)

$$(10.072) \quad \mathbf{E}' = E_0 \cos \theta \mathbf{r}_1 - E_0 \sin \theta \boldsymbol{\theta}_1 \qquad \text{for} \qquad r >> R$$

This result is shown in the diagram, Fig. 10.05. The magnitude of $\mathbf{E}'$ is

$$(10.073) \qquad E' = \sqrt{(E_0 \cos \theta)^2 + (-E_0 \sin \theta)^2} = E_0$$

and the angle θ_0 of $\mathbf{E}'$ with respect to Ox is

$$(10.074) \quad \theta_0 = \theta - \theta' = \theta - \tan^{-1}(\tan \theta) = \theta - \theta = 0$$

so that $\mathbf{E}' = \mathbf{E}_0$ as desired.

* See Appendix C.

The solution of Eq. (10.068) consistent with Eqs. (10.069) and (10.070) is

$$(10.075) \qquad V = -E_0 r \cos\theta \left(1 - \frac{R^3}{r^3}\right) + V_0 \qquad \text{volts}$$

This result can be verified by substituting it in Eq. (10.068); there is, however, no series of logical rules, *i.e.*, no analytic process for obtaining this result. It may be looked upon as a guess or, better, as the result of one of the many special solutions of Laplace's equation worked out by mathematicians and mathematical physicists during the past century.
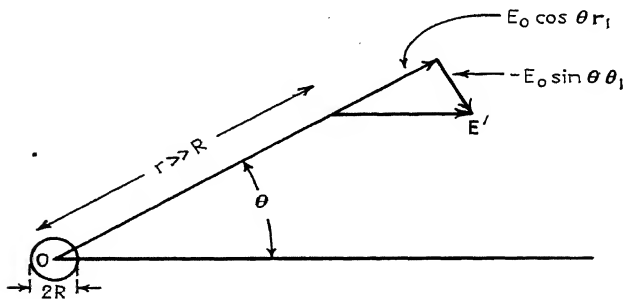
FIG. 10.05.

Note that the vector **E** can be obtained by substituting Eq. (10.075) in Eq. (10.071). Since both terms of Eq. (10.071) involve derivatives of V, the term V_0 of Eq. (10.075) does not appear in **E**. It is often convenient to choose arbitrarily the potential of a point or surface as zero. In many practical problems the potential of the earth is taken as zero, and all other potentials are measured in volts as the differences of potential between such points and earth. Thus, most calculations involve only *differences of potential* and not "absolute potentials," a phrase that is probably meaningless.

At the surface of the conducting sphere, Fig. 10.04, the electric-field intensity $\mathbf{E}_R$ is perpendicular to the surface of the sphere, *i.e.*, there is no tangential component of **E**. The magnitude of E_R can be obtained by substituting $r = R$ in

$$(10.076) \quad E_r = -\frac{\partial V}{\partial r} = \quad \text{) } \cos\theta + \qquad \text{volts per meter}$$

so that E_R is

(10.077) $E_R = 3E_0 \cos \theta$ volts per meter

Note that the electric field is zero at points A, B in Fig. 10.04,
which means that the field is zero along the great circle through
A, B whose plane is perpendicular to the paper.

Since **E** is perpendicular to the surface of a conductor, the flux
of **E** near an element of surface d**S** is

(10.078) $\mathbf{E}_R \cdot \mathbf{dS} = E_R \, dS = \dfrac{dS}{\epsilon_0}$

where σ is the average surface charge density on dS.   Therefore
the charge is distributed over the surface of the conducting sphere
so that

(10.079) $\sigma = 3\epsilon_0 E_0 \cos \theta$ coulombs per square meter

It follows that the maximum surface-charge density $(3\epsilon_0 E_0)$ is on
the element of surface surrounding the point C, Fig. 10.04; the
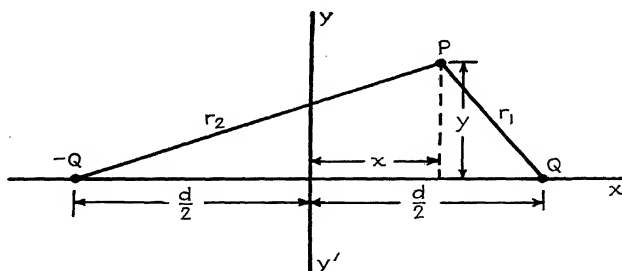
Fig. 10.06.

minimum surface-charge density $(-3\epsilon_0 E_0)$ is on the element of
surface surrounding the point D, Fig. 10.04; and there is no
surface-charge density on the elements of surface surrounding
the points A and B ($\theta = \pm 90°$).

The complete three-dimensional solution of this problem can
be represented by a figure formed by rotating Fig. 10.04 about
the line $DOCx$.

The last problem discussed in this section has to do with the
concept of *electrical images*.   In Fig. 10.06 two equal and opposite
charges are separated by a distance d.   The potential at point P

is, from Eq. (10.066),

$$(10.080) \qquad V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \qquad \text{volts}$$

Any point in the plane perpendicular to the paper through yy' is equidistant from the two charges. The plane is, therefore, from Eq. (10.080), an equipotential surface of potential zero. Since this plane is an equipotential surface, an infinite conducting sheet can be inserted coincident with the plane without disturbing the field. If such a sheet is inserted, one of the charges can be removed without disturbing the field associated with the charge on the *other* side of the sheet.

It follows that a charge Q placed near a conducting plane produces a field equivalent to that produced without the conducting plane by that charge Q together with a charge $-Q$ of like magnitude and opposite sign placed in the position of the mirror image of Q in the plane. This *principle of images* is useful for the solution of such problems as the calculation of the capacitance between a wire parallel to the surface of the earth, and the surface of the earth; it is assumed that the capacitance of wire and earth is equal to the capacitance of the wire and its image.

The electric field and the charge distribution on the plate shown in Fig. 10.06 for a point charge Q are calculated as follows:

$$(10.081) \qquad V = \frac{Q}{4\pi\epsilon_0}\left\{\frac{1}{\left[\left(\frac{d}{2}-x\right)^2+y^2\right]^{\frac{1}{2}}} - \frac{1}{\left[\left(\frac{d}{2}+x\right)^2+y^2\right]^{\frac{1}{2}}}\right\}$$

Since $\mathbf{E} = -\boldsymbol{\nabla}V = -\left(\mathbf{i}\,\dfrac{\partial V}{\partial x} + \mathbf{j}\,\dfrac{\partial V}{\partial y}\right)$, the two components of $\mathbf{E}$ are

$$(10.082) \qquad E_x = -\frac{\partial V}{\partial x} =$$

$$\frac{-Q}{4\pi\epsilon_0}\left\{\frac{\frac{d}{2}-x}{\left[\left(\frac{d}{2}-x\right)^2+y^2\right]^{\frac{3}{2}}} + \frac{\frac{d}{2}+x}{\left[\left(\frac{d}{2}+x\right)^2+y^2\right]^{\frac{3}{2}}}\right\} \text{ volts per meter}$$

$$E_y = -\frac{\partial V}{\partial y} =$$

$$\frac{-Q}{4\pi\epsilon_0}\left\{\frac{y}{\left[\left(\frac{d}{2}-x\right)^2+y^2\right]^{\frac{3}{2}}} - \frac{y}{\left[\left(\frac{d}{2}+x\right)^2+y^2\right]^{\frac{3}{2}}}\right\} \text{ volts per meter}$$

It follows from Eq. (10.082) that *along the surface* of the conducting plane ($x = 0$) the electrostatic-field intensity is everywhere zero, *i.e.*, $E_y = 0$. This follows also from the discussion above of the fact that the field is always perpendicular to the surface of a charged conductor in an electrostatic field.

Over the surface of the conductor ($x = 0$) the component of the field E_{x0} is

$$(10.083) \qquad\qquad E_{x0} = \frac{-Q}{4\pi\epsilon_0} \frac{d}{\left[\dfrac{d^2}{4} + y^2\right]^{\frac{3}{2}}} \qquad \text{volts per meter}$$

The surface charge density σ is related to E_{x0} by Eq. (10.078). Since the problem has cylindrical symmetry about the axis of x, a ring of width dy of radius y has uniform charge density σ

$$(10.084) \qquad\quad \sigma = \frac{-Q}{4\pi} \frac{d}{\left[\dfrac{d^2}{4} + y^2\right]^{\frac{3}{2}}} \qquad \text{coulombs per square meter}$$

The total charge on this ring is $2\pi y\sigma\, dy$; the total charge on the plate is

$$(10.085) \qquad \int_0^\infty 2\pi y\sigma\, dy = -\frac{Q\,d}{2} \int_0^\infty \frac{y\, dy}{\left[\dfrac{d^2}{4} + y^2\right]^{\frac{3}{2}}}$$

$$= -\frac{Q\,d}{2} \left[\frac{-1}{\left(\dfrac{d^2}{4} + y^2\right)^{\frac{1}{2}}} \right]_0^\infty = -Q$$

Thus the charge on the conducting plate is equal in magnitude to the "point" charge Q, but opposite in sign.

This concludes an introductory discussion of electrostatic problems involving distributions of charges in a vacuum, or, to a close approximation, in air. When solid or liquid dielectrics are present in the region in which the charges are distributed, the theory developed and applied above is inadequate. Additional concepts, required for solving such problems, are discussed in the following sections.

10.11. The Macroscopic Point of View Relative to Electrostatic Fields.—The relations thus far developed in this chapter are based on the assumption that at any instant of time one may assign to the electric-charge density at any point, either in free space or in a material medium, a definite value ρ, which is a

function of the coordinates x, y, and z of this point. These relations are also based on the assumption that ρ and its first- and second-order partial derivatives with respect to x, y, and z are finite and single-valued functions of these coordinates. This second assumption, though probably correct, is not essential to the mathematical coordination of the physical phenomena usually referred to as "directly observable" electrostatic phenomena. For example, the mechanical force exerted by one charged body on another, when their distance apart is large relative to the linear dimensions of these bodies, and the intervening medium is air, is

$$(10.086) \qquad\qquad F = \frac{1}{4\pi\epsilon}\frac{QQ'}{r^2} \qquad\qquad \text{newtons}$$

where Q and Q' are respectively the *total net charge* on these two bodies. In fact, it was in this form that Coulomb stated the results of his experiments on charged bodies, and for this reason Eq. (10.086) is frequently referred to as Coulomb's law, as noted in Sec. 10.03.

More generally the observed phenomena attributable to the positions and motions of electric charges justify the following assumptions:

1. In its normal state a molecule of a substance contains equal positive and negative charges that are in general in motion relative to each other.

2. A part of the positive electricity in each molecule of a substance is *bound* to the negative electricity in the same molecule in such a manner that neither can pass from one molecule to another without rupturing the given substance. Such charges will be referred to as *bound charges*.

3. Some of the electric charges in the molecules of a substance may move freely from molecule to molecule in this substance without rupturing it. Such charges will be referred to as *free charges*.

Substances in which the free charges (even though relatively few in number) play the predominant role are called *conductors*. Substances in which the bound charges play the predominant role are called *dielectrics*, or *insulators*. A vacuum, or free space, is also called a *dielectric*. Substances in which both types of charges play an important role are called *semiconductors*.

The concepts of electrostatic potential and electrostatic intensity above developed are applicable* at every instant to every point in space whether or not this point is in an electric charge or in a neutral particle of matter.   However, the precise application of these concepts to ordinary observable phenomena would involve the consideration of the exact value of the electric-charge density at every instant at every point in each conductor and in each dielectric in the electrostatic field under consideration.   For many purposes such precision is entirely unnecessary, and one may instead coordinate the experimental facts by mathematical relations based on a consideration of the *average* positions of the electric charges in contiguous *elementary* volumes, Δv_n, each so large that it contains many (even millions of) molecules and yet so small that it may be considered mathematically as an infinitesimal.   Such an elementary volume Δv_n will be referred to as a *macroscopic infinitesimal volume.*   The relations based upon the consideration of such a volume as an infinitesimal in the strict mathematical sense will be referred to as the *macroscopic theory of electricity.*

Designate by ρ the actual charge density at any point in a macroscopic volume Δv_n at any instant of time, as defined by Eq. (10.016) of Sec. 10.04.   Then the total net charge in this volume Δv_n is

$$(10.087) \qquad \qquad \mathcal{Q}_n = \int_{\Delta v_n}$$

where dv is an infinitesimal in the strict mathematical sense; see Eq. (10.018) of Sec. 10.04.   The average value $\bar{\rho}_n$ of the net charge density in the elementary volume Δv_n, at the specified instant of time, is then

$$(10.088) \qquad \qquad \bar{\rho}_n = \frac{1}{\Delta v}$$

If this average value $\bar{\rho}_n$ remains constant for a specified interval of time $(t_2 - t_1)$, then the net electric charge in the macroscopic infinitesimal volume Δv_n is said to remain *static* during this interval.

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* Even when the charges are in motion, provided a slight modification is made in the formula, Eq. (10.019), for the absolute electric potential V; see Sec. 16.05 and Appendix E.

The average value $\bar{\rho}$ of the electric charge in a macroscopic infinitesimal volume Δv may always be expressed in the form

(10.089) $$\bar{\rho} = \bar{\rho}_f + \bar{\rho}_b$$

where $\bar{\rho}_f$ is the average value of the free charge density in the volume Δv and $\bar{\rho}_b$ is the average value of the bound charge density in this volume.

Experiment justifies the assumption that wherever two different media touch each other this touching results in the formation of an extremely thin layer with two "faces," one of which lies in one medium and the other in the second medium, and that in the volume between these two faces each medium is "absorbed" to a greater or less extent in the other.   In other words, there is
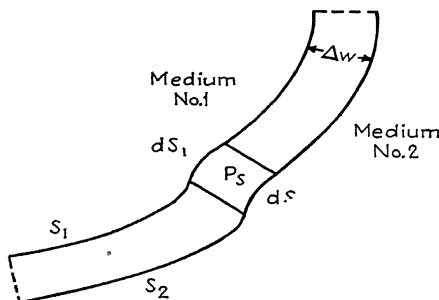
Fig. 10.07.

actually no geometrical surface that forms a definite demarkation between two media in contact, but rather a merging of one medium into the other.   Within the contact layer this merging is probably a gradual, or continuous, change in the nature of the medium from one of the faces of this layer to the other.

Referring to Fig. 10.07, let S_1 and S_2 be the two surfaces that form the two faces of the contact layer between a medium No. 1 and another medium No. 2.   Let Δv be an elementary right cylinder whose two ends ΔS_1 and ΔS_2 lie in these two surfaces and whose height is Δw, and let P_s be a point in Δv.   In the limit when Δw is considered an infinitesimal in the strict mathematical sense, the volume Δv is equal to $\Delta S \Delta w$, where ΔS is the infinitesimal surface to which both ΔS_1 and ΔS_2 tend as Δw tends to zero.   Designate by ΔQ_f and ΔQ_b the free charge and the bound

charge in this volume Δv.   The two ratios

(10.090)

(10.091) $$\frac{\Delta Q_b}{\Delta S} = \bar{\rho}_b \, \Delta w$$

are called respectively the *surface density of the free charge* and the *surface density of the bound charge* at the element dS of the *contact surface* between the two media.   The algebraic sum of the surface densities defined by Eqs. (10.090) and (10.091), *viz.*,

(10.092) $$\sigma = \sigma_f + \sigma_b$$

is called the surface density of *the resultant electric surface-charge density at dS*.

It should always be kept in mind that the term "contact surface" as here used is not a surface in the strict geometrical sense, but is actually a layer of elementary thickness Δw.   From the macroscopic point of view this thickness Δw is treated as an infinitesimal dw, and the areas ΔS_1 and ΔS_2 in the two faces of this contact layer are each treated as the infinitesimal area dS to which each of the areas ΔS_1 and ΔS_2 tend as Δw tends to zero.

The electric phenomena that can be coordinated by the macroscopic theory are those in which one is justified, by experiment, in assuming that in a system of conductors and dielectrics the average charge density, both free and bound, is zero in every elementary volume Δv_n *except those which lie, in whole or in part, in a contact layer between two different media,* or which lie in the so-called *space charge* that can be produced in a vacuum.   In the following discussion, unless specifically stated otherwise, space charges will be ignored.

When only the electric charges in the contact layers between conductors and dielectrics need be considered, the average electrostatic potential in any elementary volume that contains a point P not *in a contact layer* may be formulated as follows.   Let r be the distance of this point from an infinitesimal area dS in the surface that corresponds to this contact layer, and let σ be the surface density of the resultant charge at this surface, as defined by Eq. (10.092).   Then the function

(10.093) $$V = \qquad \sigma \, dS$$

where S is the total surface of contact between all different media, is *defined* as the average electrostatic potential in the elementary volume that contains the point P. To distinguish this from the function defined by Eq. (10.019) of Sec. 10.04, this average electrostatic potential will be referred to as the *macroscopic* (big-view) *electrostatic potential*. The electrostatic potential defined by Eq. (10.019) of Sec. 10.04 may then be thought of as the *microscopic* (little-view) *electrostatic potential*.

The macroscopic electrostatic potential defined by Eq. (10.093) is finite, single-valued, and continuous for all points, even for those which lie in a contact surface. The first- and second-order partial derivatives of this function with respect to the coordinates that define the position of the point P relative to any arbitrarily chosen system of coordinates are likewise finite, single-valued, and continuous *except at points where σ has a value different from zero*. Hence, at every point at which $\sigma = 0$ the macroscopic electrostatic potential defined by Eq. (10.093) has a gradient, which gradient is the negative of the *macroscopic electrostatic intensity* at this point, *viz.*,

$$(10.094) \qquad \mathbf{E} = -\boldsymbol{\nabla} V = -\left(\mathbf{i}\, \frac{\partial V}{\partial x} + \mathbf{j}\, \frac{\partial V}{\partial y} + \mathbf{k}\, \frac{\partial V}{\partial z} \right)$$

Also, *at every point at which $\sigma = 0$* the divergence of this macroscopic electrostatic intensity is zero, *viz.*,

$$(10.095) \qquad\qquad \boldsymbol{\nabla} \cdot \mathbf{E} = -\nabla^2 V = 0$$

When the surface density σ at point P is not zero, then the first-order partial derivatives of the macroscopic electrostatic potential V are in general double-valued, and the second-order partial derivatives of this electrostatic potential may be multi-valued or infinite. Therefore the divergence of the macroscopic electrostatic intensity $\mathbf{E}$ defined by Eq. (10.094) has no significance at a point at which the resultant surface density of the electric charge is different from zero. However, at a contact surface, where σ is in general not equal to zero, one may establish certain extremely useful *boundary conditions*. This is done in the following manner. Consider a point P_s in a contact surface, and let dS be an infinitesimal area of this surface at this point. Imagine a normal drawn through dS in the sense from medium No. 1 to medium No. 2. Let P_1 and P_2 be two points on this

normal, one in each medium, and each at an infinitesimal distance from P_s. Let E_1 and E_2 be the magnitudes of the macroscopic electrostatic intensities at P_1 and P_2 respectively and let θ_1 and θ_2 be the angles made by the directions of these two intensities with the specified normal. Then, as P_1 and P_2 approach coincidence with P_s, the flux of the vector **E** *into dS* from the side on which P_1 is located tends to the value $(E_1 \cos \theta_1)\, dS$, and the flux of **E** *away from dS* on the side on which P_2 is located tends to the
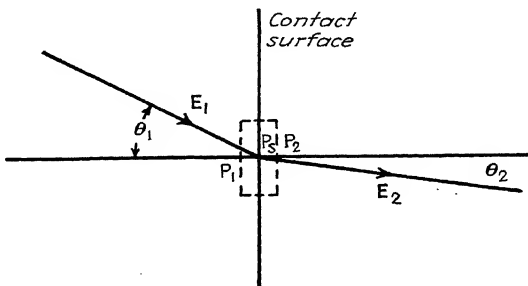
Fɪɢ. 10.08.

value $(E_2 \cos \theta_2)\, dS$. The net outward flux of **E** from the surface dS then tends to the value

$$(10.096) \qquad (E_2 \cos \theta_2 \qquad \cos \theta_1)\, dS$$

For this relation to be consistent with the microscopic theory this outward flux (from Gauss's theorem) must be equal to the resultant charge at the contact surface dS divided by ϵ_0, *viz.*,

$$(10.097) \qquad \frac{\sigma\, dS}{\epsilon_0}$$

Equate (10.096) and (10.097) and divide by dS; there results

$$(10.098) \qquad E_2 \cos \theta_2 = \qquad \cos \theta_1 + \frac{\upsilon}{\epsilon_0}$$

On the other hand, since the macroscopic electrostatic potential is single-valued at *every* point, including points on a contact surface, the circulation of the corresponding electrostatic intensity **E** around the path indicated by the dotted lines in Fig. 10.08 must be zero, no matter how close the "sides" of this path are to the

contact surface. Hence in the limit, when these two sides coincide with the contact surface,

$$(10.099) \qquad E_2 \sin \theta_2 = E_1 \sin \theta_1$$

This relation is usually described by the statement that the *tangential components* of the macroscopic electrostatic potential on the two sides of *every surface* are *equal*.

From Eq. (10.098) the total flux of the macroscopic electrostatic intensity **E** outward from both sides S_1 and S_2 of a surface S (see Fig. 10.08) is

$$(10.100) \qquad \int_{S_2} E_2 \qquad\qquad\qquad = \frac{1}{\epsilon_0} \int_S \sigma \, dS$$

The integral in the extreme right-hand member of this expression is equal to the algebraic sum of the free charge Q_f and the bound charge Q_b in the contact layer whose two faces are S_1 and S_2, *viz.*,

$$(10.101)$$

The difference between the two integrals in the middle member may be represented by the symbol

$$(10.102) \qquad \int_{S_1 S_2} \mathbf{E} \cdot \mathbf{dS}$$

where $\mathbf{E} \cdot \mathbf{dS}$ is the dot product of **E** and **dS** at each element of the complete bounding surface formed by the two faces S_1 and S_2. Therefore Eq. (10.100) may be written

$$(10.103) \qquad = \int_{S_1 S_2} \mathbf{E} \cdot \mathbf{dS} = \frac{1}{\epsilon_0}$$

The determination of the bound charge at a contact surface is not readily effected by any direct experimental procedure. However, such a charge at a contact surface may be expressed in terms of an auxiliary vector

$$(10.104) \qquad \mathbf{D} = \epsilon \mathbf{E}$$

where **E** is the electrostatic intensity at any point P that is not in the contact layer between two different media, and the factor ϵ is a measurable property of the medium in which P is located. This vector **D** is called the *density of the electrostatic induction* at

the point P, or, more briefly, the *electrostatic-flux density* at this point, and the factor ϵ is called the *electric permittivity* of the medium at this point, or, more briefly, the *permittivity* at this point.

In the defining Eq. (10.104) the permittivity ϵ is so chosen that the *total outward flux of the vector D from every macroscopic volume Δv that contains a free charge is identically equal to the total FREE CHARGE within this volume.* This condition requires that if the net *free charge* in any specified layer of thickness Δw is zero, then the net outward flux of $\mathbf{D}$ *into* this layer on one side is equal to the net outward flux of $\mathbf{D}$ *from* the other side of this layer. On the other hand, if such a layer contains a net free charge Q_f, then the flux of $\mathbf{D}$ outward from both sides of this layer is Q_f, *viz.*,

$$(10.105)^* \qquad \psi_D = \int_{S_1 S_2} \mathbf{D} \cdot \mathbf{dS} = Q_f$$

In this expression the symbol $S_1 S_2$ at the base of the integral sign has the same significance as in Eq. (10.103), *viz.*, that the surface over which the integration is to be performed is the bounding surface formed by the two faces of the layer under consideration (see Fig. 10.07). If the net free charge Q_f is zero, then the net outward flux ψ_D defined by Eq. (10.105) is also zero, even though in this layer the net bound charge Q_b has a value different from zero.

Designate by D_1 and D_2 respectively the magnitudes of the vector $\mathbf{D}$ at the infinitesimal plane areas dS_1 and dS_2 in the faces of the contact layer between two media (see Fig. 10.07) and designate by α_1 and α_2 the angles made by the direction of this vector with the normal drawn through a point P_s in the contact layer in the sense from medium No. 1 to medium No. 2. Then the flux of the vector $\mathbf{D}$ *outward* from the contact layer into medium No. 2 is

$$(10.106) \qquad \psi_{D_2} = \int_{S_s} D_2 \cos \alpha_2 \, dS$$

---

* Note from Eq. (10.056) that the differential equation corresponding to the integral form, Eq. (10.105), is $\nabla \cdot \mathbf{D} = \rho$. However, within those regions in which $\mathbf{D}$ is discontinuous, $\nabla \cdot \mathbf{D}$ does not exist. Therefore, values of $\mathbf{D}$ *on the surfaces* of such regions are used as *boundary conditions* for the differential equation $\nabla \cdot \mathbf{D} = \mu$, which has meaning only for those regions in which $\mathbf{D}$ *is* continuous.

and the flux of **D** *inward* to this layer from medium No. 1 is

$$(10.107) \qquad \psi_{D_1} = \qquad \cos \quad \iota \, dS$$

Equation (10.105) may then be written

$$(10.108) \qquad \psi_{D_2} - \psi_{D_1} = Q_f$$

Consider first the case when the permittivity ϵ of each medium is a constant scalar, say ϵ_1 for medium No. 1 and ϵ_2 for medium No. 2. Under these conditions the directions of $\mathbf{D}_1$ and $\mathbf{D}_2$ are respectively the same as the directions of $\mathbf{E}_1$ and $\mathbf{E}_2$, and therefore $\alpha_1 = \theta_1$ and $\alpha_2 = \theta_2$, and Eqs. (10.106) and (10.107) may be written

$$(10.109) \qquad \frac{\psi_{D_2}}{\epsilon_2} = \int_{S_1} E_2 \cos \theta_2 \, dS$$

$$(10.110) \qquad \frac{\psi_{D_1}}{\epsilon_1} = \int_{S_2} E_1 \cos$$

Therefore from Eqs. (10.100) and (10.103) the bound charge in the layer whose faces are the two surfaces S_1 and S_2 is

$$(10.111) \qquad Q_b = -\epsilon \qquad \epsilon_0 \qquad \epsilon_0$$

From Eq. (10.105), *if the net free charge in the layer between the two faces S_1 and S_2 is zero,* then the flux ψ_{D_2} is equal to the flux ψ_{D_1}, and therefore Eq. (10.111) may be written

$$(10.112) \qquad = \left(\frac{\epsilon_0}{\epsilon_2} - \frac{\epsilon_0}{\epsilon_1} \right)$$

where ψ_{D_2} is the electrostatic flux through this layer in the sense from medium No. 1 to medium No. 2. If the layer under consideration is wholly within a homogeneous medium ($\epsilon_1 = \epsilon_2$) or if this layer is the contact layer between two media that have the same permittivity, then the net bound charge Q_b in this layer is *zero*.

Experiment justifies the assumption that in a *conducting* medium in which there is no macroscopic flow of electricity and no electromotional intensity (see Sec. 12.03), the macroscopic electrostatic intensity is *zero*. Consider the contact layer where a conductor and a dielectric touch each other, and assume these

"static" conditions to hold. The electrostatic flux density into this layer from the conductor side is zero, *viz.*,

$$(10.113) \qquad \psi_{D_1} = 0$$

Consequently, from Eq. (10.108)

$$(10.114) \qquad \psi_{\Gamma_2} = Q_f$$

and from Eq. (10.111) the bound charge at this surface is

$$(10.115) \qquad Q_b = - \left(1 - \frac{\epsilon_0}{\epsilon} \right.$$

where ψ_{D_2} is the electrostatic flux into the dielectric through the surface under consideration. Note particularly that if ϵ_2 is greater than ϵ_0, then the bound charge at such a surface is *negative* when the free charge Q_f is positive, and positive when Q_f is negative.

In the above expressions for the bound charge it is the *ratio* of the permittivity of a given medium to the permittivity of free space that determines whether or not the net bound charge in a given layer is different from zero. This ratio, *viz.*,

$$(10.116) \hspace{6cm} \text{(dimensionless)}$$

where ϵ is the permittivity of a given medium and ϵ_0 is the permittivity of free space, is called the *dielectric constant* of the given medium. For homogeneous media this constant is readily determined.* For a vacuum it is unity. For air and other gases at ordinary temperatures and pressures it differs from unity by less than 0.1 per cent. The approximate values of the dielectric constant for a few solids and liquids are given in the table on page 268.

The electrostatic-flux density **D** at any point must not be confused with the vector **E**, which is the flux per unit area of the electrostatic intensity. In the mks system of units these two quantities are never equal, even in free space, and irrespective of the system of units employed, the dimensions of **D** and **E** are always different, although in the cgs electrostatic system of units the magnitudes of these two vectors are numerically equal when the dielectric constant $k = \epsilon/\epsilon_0$ is unity.

* See Sec. 11.05.

In the case of certain crystals the factor ϵ in expression $\mathbf{D} = \epsilon\mathbf{E}$ is not a constant, but its value depends upon the relative directions of the vector $\mathbf{E}$ and the axes of the crystal.   Such anisotropic media will not be considered here.

Material	Approximate Dielectric Constant at 18°C.
Amber..	2.9
Asbestos paper................................	2.7
Bakelite......................................	4.5
Glass...	5 to 10
Mica..	3.5 to 10
Shellac.......................................	3
Sulphur.......................................	3.5
Dry wood......................................	3–6
Ethyl alcohol.................................	23
Acetone.......................................	21
Carbon bisulphide.............................	2.6
Ethyl ether...................................	4.3
Glycerin......................................	25
Castor oil....................................	4.7
Petroleum oil.................................	2.2
Water...	82

When two different isotropic dielectrics are in contact at a given surface S and there is no free charge at this contact surface, both the macroscopic electrostatic intensity $\mathbf{E}$ and the electrostatic-flux density $\mathbf{D}$ in general undergo an abrupt change in direction at this contact surface.   This may be shown as follows. From Eq. (10.099)

(10.117) $$E_2 \sin \theta_2 = E_1 \sin \theta_1$$

and from the fact that there is no free charge at this surface

(10.118) $$D_2 \cos \theta_2 = D_1 \cos \theta_1$$

Hence

$$\frac{\overline{\tan \theta_1}}{\tan \theta_2} = \frac{D_1}{E_1}\frac{E_2}{D_2} = \frac{k_1}{k_2}$$

Therefore, unless θ_2 is zero or $k_1 = k_2$, there is an abrupt change in the direction angle θ from θ_1 to $\tan^{-1}[(k_2/k_1)\tan\theta_1]$ as the flux passes through the given surface.

10.12. Electrostatic Field of Uniformly Charged Sphere.—As a simple illustration of some of the concepts developed in Sec.

10.11, consider a spherical surface of radius R on which the charge density $\sigma = \sigma_f + \sigma_b$ has the same value at every point. The total charge on this surface is then

(10.120) $$Q = 4\pi R^2 \sigma$$

Referring to Fig. 10.09 let a be the distance of any point P from the center O of this sphere and let P_s be a point in the spherical surface at a distance r from P. Let θ be the angle between the two lines OP and OP_s, and let $(\theta + d\theta)$ be the angle between OP and the line OP'_s to a point P'_s in the spherical surface at the
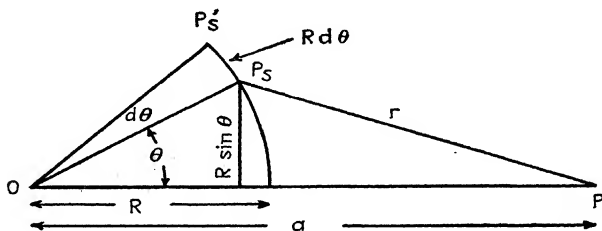
FIG. 10.09.

infinitesimal distance $R\,d\theta$ from P_s. Imagine two planes perpendicular to OP, one passing through the point P_s and the other through the point P'_s. Then every point in the zone of width $R\,d\theta$ in the spherical surface may be considered as being at the same distance r from P. The area of this zone is $(2\pi R \sin\,\theta)$ $(R\,d\theta)$. Therefore from Eq. (10.066) the potential at P due to the charge in this zone is

(10.121) $$\sigma R^2 \sin\,\theta\,d\theta$$

Note that for any position of P_s

(10.122) $$r^2 = a^2 + R^2 - 2\,aR\,\cos\,\theta$$

and therefore, by differentiation, holding a and R constant,

(10.123) $$r\,dr = aR\,\sin\,\theta\,d\theta$$

Consequently

(10.124) $$\frac{\sin\,\theta\,d\theta}{r} = \frac{dr}{aR}$$

This substituted in Eq. (10.121) gives

$$(10.125) \qquad dV = \frac{\sigma R \, dr}{2a\epsilon_0}$$

When P is *outside* the spherical surface the limits of r are respectively $(a - R)$ and $(a + R)$, and therefore the total potential at P due to the entire charged spherical surface is

$$(10.126) \qquad V_0 = \frac{\sigma R^2}{a\epsilon_0} \qquad \text{volts}$$

which from Eq. (10.120) may also be written

$$V_o = \frac{1}{\quad} \, \underline{Q} \qquad \text{volts}$$

Note that the radius R does not appear in this final formula. Therefore, for all points *exterior* to a uniformly charged spherical surface of radius R the electrostatic potential due to the total charge Q on this surface is identical with the electrostatic potential that would be produced at this point were this total charge confined to a spherical surface of any *smaller* radius, even though this smaller radius be an infinitesimal. For this reason, the electrostatic potential *external* to a uniformly charged spherical surface on which the total charge is Q is said to be identical with the electrostatic potential that would be produced at this point by a *point charge Q* located at the center of this sphere.

When P is *inside* the spherical surface the limits of r are $(R - a)$ and $(R + a)$ and therefore the total potential at P due to the entire charged surface is, from Eq. (10.125)

$$(10.128) \qquad\qquad \epsilon_0 \qquad\qquad \text{volts}$$

which from Eq. (10.120) may also be written

$$(10.129) \qquad\qquad V_i = \quad \frac{1}{\quad} \, \underline{Q} \qquad\qquad \text{volts}$$

Note that in this expression the distance a of the point P from the center of the sphere does not appear, but V_i for a given charge Q uniformly distributed over a spherical surface of a given radius R is a *constant*, equal to the potential that a *point charge Q* located at the *center* of this sphere would produce at the *surface*

of this sphere. Therefore, not only is the spherical surface an equipotential surface but the entire volume of the sphere is an equipotential volume.

From Eq. (10.127) the electrostatic intensity at any point *external* to the charged spherical surface under consideration has the magnitude

$$(10.130) \qquad E_e = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \qquad \text{volts per meter}$$

whereas, from Eq. (10.129), the electrostatic intensity at every *internal* point is zero, *viz.*,

$$(10.131) \qquad E_i = 0$$

At the spherical surface the electrostatic potential tends to the same value, irrespective of whether this surface is approached from an internal or an external point; therefore V is a *continuous* function of the space coordinates. On the other hand, the electrostatic intensity remains zero inside the surface no matter how closely this surface is approached, but outside the electrostatic intensity tends to the nonzero value given by Eq. (10.130); therefore the electrostatic intensity has a *discontinuity* at the surface on which the charge Q is located.

In the above deductions no assumptions were made in regard to the nature of the medium inside the spherical surface or outside this surface. The formulas are all valid irrespective of the nature of these two media provided no charges (free or bound) other than those at the spherical surface manifest themselves anywhere within a finite distance of the center of the sphere. For example, the surface in question may be the contact surface between a conducting sphere and a homogeneous dielectric that has a dielectric constant k and that extends to "infinity." Under these conditions the total charge Q is equal to the algebraic sum of the free charge Q_f on the conductor (carried there, for example, by the flow electricity from some other body) and the bound charge Q_b at the contact surface between the sphere and the surrounding dielectric. In this case

$$(10.132) \qquad Q = Q_f + Q_b$$

where Q_b is given by Eq. (10.115), *viz.*,

$$(10.133) \qquad = -\left(1 - \frac{1}{k}\right) Q_f \qquad \text{coulombs}$$

and therefore

$$(10.134) \qquad\qquad Q = \frac{Q_f}{k} \qquad\qquad \text{coulombs}$$

The macroscopic electrostatic potential at any point in the dielectric at a distance $a > R$ from the center of the sphere is then

$$(10.135) \qquad\qquad \frac{1}{} \quad Q_f \qquad\qquad \text{volts}$$

and the macroscopic potential at every point at a distance $a < R$ from the center of the sphere is

$$(10.136) \qquad\qquad V_i = \frac{1}{4\pi\epsilon_0} \frac{Q_f}{kR} \qquad\qquad \text{volts}$$

In terms of the *free charge alone* the potential both outside and inside the conducting sphere is then inversely proportional to the dielectric constant of the surrounding medium. This concept, frequently introduced in elementary textbooks, is valid only if the dielectric medium that surrounds the charged conductors under consideration is *homogeneous and has no bound charges at any boundary other than those at the contact surface, or surfaces, between it and these charged conductors.* Equations (10.135) and (10.136) are not valid when the surrounding dielectric is made up of two or more media of different dielectric constants.

10.13. Dielectric Polarization.—The macroscopic electrostatic potential V is defined in Sec. 10.11 and used there as the basis for the development of the theory of electrostatics in material bodies. The practical value of this theory is that, by introducing the vector $\mathbf{D} = \epsilon\mathbf{E}$, electrostatic problems can be solved in terms of *free* charges, while bound charges need not be explicitly considered.

The conclusions reached in Sec. 10.11 can be derived also from certain assumptions concerning the properties of insulating materials, expressed in terms of a vector $\mathbf{P}$ known as the *dielectric polarization;* this is the subject of this section.

According to the theory discussed in Chap. I, "homogeneous" material bodies consist of molecules that are electrically neutral. In the molecules of electrical insulators no electric field of magnitude lower than that which would cause disruption of the material

can cause electrons to move a large enough distance from their respective atomic nuclei to become independent of the binding forces of these nuclei. However, if an electric field is applied to the dielectric, the electrons of a molecule may acquire, according to this theory, a small displacement opposite to the applied field and the positive atomic nuclei may acquire a small displacement in the direction of the electrostatic field. If some such phenomena occur, the average volume charge density would remain constant (zero) within a homogeneous material but *the surface charge densities would depend upon the magnitude and direction of the electrostatic field.* Some of these hypothetical properties have been subjected to indirect tests.[7]

If the positive and negative charges of a molecule can be slightly displaced by an electrostatic field, an idealized representa-
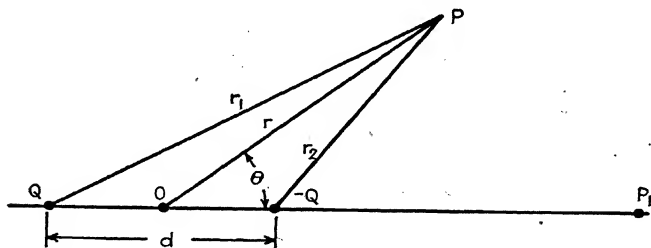
FIG. 10.10.

tion of them would be an *electrostatic dipole*, shown in Fig. 10.10. The potential V at P is

(10.137) $$V = Q \left(\frac{1}{} \quad \frac{1}{} \right)$$ volts

If Q, $-Q$ represent the charges, and d the small displacement of the charges of the molecules of a dielectric, most problems are concerned with the circumstances for which $r >> d$. By substituting for r_1 and r_2 in Eq. (10.137) in terms of r and θ, and neglecting terms involving d^2/r^2, Eq. (10.137) reduces to the approximate relation

(10.138) $$V = \frac{Qd \cos \theta}{}$$ volts

This result is commonly written in terms of the *moment* of the dipole. The *moment* **p** of a dipole is defined as a *vector* having a

magnitude equal to the product of the magnitude of the charge Q and the distance of separation d, and a direction parallel to the line from $-Q$ to $+Q$.

$$(10.139) \qquad \mathbf{p} = Q\mathbf{d}$$

Thus, representing a unit vector parallel to OP of Fig. 10.10 by $\mathbf{r}_1$, (10.138) becomes

$$(10.140)^* \qquad V = \frac{1}{4\pi\epsilon_0 r^2}\, \mathbf{p} \cdot \mathbf{r}_1$$

The incorporation of this concept of the dipole into a theory of dielectrics is worked out in the paragraphs below for the following practical reasons. Consider a condenser, consisting of two metallic electrodes separated by a volume in which there are one or more homogeneous dielectric materials. When a difference of potential exists between the metal electrodes, this difference of potential and the free charges ($+$ or $-$) on the electrode *can be measured*. Electrons and ions of the one or more dielectrics are, according to the theory described above, displaced from their normal positions, *i.e.*, from their positions when the difference of potential between the electrodes is zero. This produces bound surface–charge densities σ_b on the surfaces of the dielectrics. These surface–charge densities affect the field within the condenser, but *these bound charge densities associated with the dielectrics are not directly measurable*. A practical theory of dielectric effects must relate (1) measurable properties of the dielectric materials and (2) measurable properties of the electrostatic influence that produce electrostatic fields in dielectric materials.

Assume first, following the method of Sec. 10.11, that a small volume $dv_a = dx\,dy\,dz$ within a dielectric material *not* subjected to external electrostatic influences contains (1) many molecules and (2) equal amounts of positive and negative electric charge. The assumptions are in accord with the concepts of atomic

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* Note that the potential and electrostatic-field intensity at a point such as P_1, Fig. 10.10, are

$$(10.141) \qquad V \qquad\qquad\qquad\qquad\qquad\qquad \text{volts}$$

$$(10.142) \qquad\qquad \frac{Qd}{} \qquad \text{volts per meter in the direction } -Q \rightarrow Q$$

theories discussed above and with the concept of a distribution of charge density discussed in Sec. 10.11. Assume also—quite arbitrarily—that the *negative* charges within the volume $dv_a = dx\,dy\,dz$ are fixed, *i.e.*, immovable, and that the average volume density of *positive* charge in dv_a considered independently is $\overset{+}{\rho}_a$ coulombs per cubic meter.

According to these assumptions, the application of an external electrostatic field to the dielectric material, of which dv_a is a part, causes the production of many molecular dipoles within the material. Since it is here assumed that the negative charges remain fixed, the formation of dipoles in the neighborhood of dv_a is equivalent to a displacement $\mathbf{d}$ of the charge $\overset{+}{\rho}_a\,dv_a$. The dielectric polarization $\mathbf{P}$, sometimes called the *electrostatic moment per unit volume*, is defined as

(10.143) $\mathbf{P} = \overset{+}{\rho}_a\mathbf{d}$ coulombs per square meter

in which $\mathbf{d}$ is in both magnitude and direction the average displacement assumed for the positive charges of the many molecules near the point at which the polarization is $\mathbf{P}$.

It is shown in Appendix B that the outward flux of a vector, per unit volume, from a volume dv_a is defined as the divergence of the vector at the point about which dv_a is constructed. It can be shown that the divergence of the vector $\mathbf{P}$ is the *outward* flux per unit volume of positive charge.

Since the net charge density before polarization is zero and the negative charges are assumed to be fixed, the effect of polarization is to produce a bound charge density ρ_b

(10.144) $\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P}$ coulombs per cubic meter

It is in accord with experiment to assume that the bound *volume-*charge density ρ_b *within* a homogeneous, isotropic dielectric is zero. At the surfaces, however, the effect of polarization is to produce a bound volume-charge density ρ_b that corresponds to a displacement to or from this surface of the charge $\overset{+}{\rho}_a$ by a distance $\mathbf{d}$; it is convenient to call this bound charge on the surface the *bound surface-charge density* σ_b coulombs per square meter.

A series of simple diagrams to illustrate the points discussed above is shown in Fig. 10.11. The rectangle at the top of the figure represents a uniform sheet of positive charge; the next rectangle represents a uniform sheet of negative charge. The

rectangle marked O represents the two upper rectangles superimposed; the net charge is zero, and this rectangle represents an unpolarized dielectric. If these charges are subjected to a field produced by the two charged metal electrodes A, B, and if the negative charges are fixed, the positive charges are displaced to the right as shown at the bottom of Fig. 10.11. In the bottom diagram, the displacement of each point in the positive sheet represents the polarization **P** at that point. Throughout the
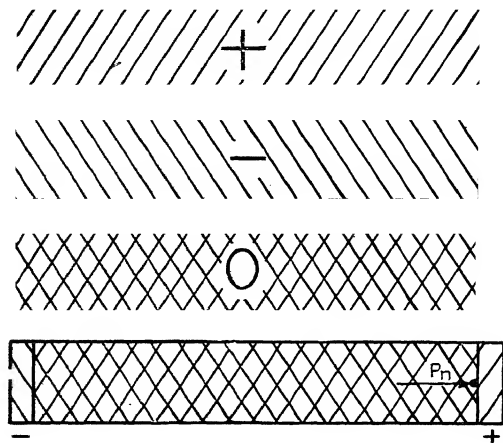
Fig. 10.11.

sheet, except at the lateral edges, the net charge density remains zero after the displacement; this corresponds to the statement that the bound volume-charge density $\rho_b = -\nabla \cdot \mathbf{P}$ is zero *within* the volume of the dielectric. Note that in this region **P** has a value but $\nabla \cdot \mathbf{P} = 0$. Along the lateral edges there are strips of charge, negative on the left, positive on the right; these correspond to bound surface charges. The bound surface-charge density σ_b at any point on one of these surfaces is equal to the normal component P_n of the polarization.

(10.145) $$\sigma_b = P_n \quad \text{coulombs per square meter}$$

in which P_n is the component of **P** *outward* from the body of the dielectric into the contact surface. The bottom diagram of

Fig. 10.11 represents a case where $\mathbf{P}$ is directed from left to right throughout the volume; the component P_n is shown in the diagram. Therefore, according to Eq. (10.145), the bound surface-charge density over the surface near B is positive, while over the surface near A it is negative.

It will now be shown that the vector $\mathbf{P}$, defined in this section in terms of hypothetical displacements of positive charges in dielectric materials, can be expressed also in terms of the vectors $\mathbf{D}$ and $\mathbf{E}$, which were defined in Sec. 10.11; this expression is

$$(10.146) \qquad \mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E} \quad \text{coulombs per square meter}$$

First note, from a generalization of Eq. (10.103), that the flux of $\mathbf{E}$ outward from a volume v_a of a surface layer between two faces S_1 and S_2 is

$$(10.147) \qquad \int_{S_1 S_2} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \int_{v_a} \rho_f \, dv_a + \frac{1}{\epsilon_0} \int_{v_a} \rho_b \, dv_a$$

in which ρ_f and ρ_b are the free and bound volume-charge densities. This relation is equivalent, from Eq. (10.144), to

$$(10.148) \qquad \int_{S_1 S_2} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \int_{v_a} \rho_f \, dv_a - \frac{1}{\epsilon_0} \int_a \nabla \cdot \mathbf{P} \, dv_a$$

The last term of Eq. (10.148) can be converted to a surface integral of $\mathbf{P}$, by Gauss's theorem, and then transferred to the left-hand side of the equation so that, when both members are multiplied by ϵ_0, there results

$$(10.149) \qquad \int_{S_1 S_2} \qquad\qquad = \int_v \rho_f \, dv = Q_f \quad \text{coulombs}$$

in which Q_f is all of the *free* charge within the surface S. Thus the vector sum $(\epsilon_0\mathbf{E} + \mathbf{P})$ is the *electric-flux density* or *displacement* $\mathbf{D}$ defined in Sec. 10.11, *viz.*,

$$(10.150) \qquad \mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \quad \text{coulombs per square meter}$$

In general $\mathbf{E}$ and $\mathbf{P}$ may have different directions; such is the case for some crystals.* However, most practical problems are con-

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* In such cases it is possible to write $\mathbf{D} = \epsilon^*\mathbf{E}$, where ϵ^* is a dyadic; see *Electromagnetic Theory;* J. A. Stratton; p. 11, McGraw-Hill Book Company, Inc., New York, 1941.

cerned with *those special cases* of the general theory discussed in this chapter for which the following assumptions are justifiable:

1. $\mathbf{P}$ is in the same direction as $\mathbf{E}$.
2. P is proportional to E.

These assumptions are equivalent to the hypothesis that, when the molecules in a dielectric are subjected to an electric field, the resulting separation of their positive and negative electricity (to form dipoles) is in the direction of the applied field and proportional to the magnitude of the applied field, regardless of the direction of the applied field with respect to axes fixed with respect to the *dielectrics*. These assumptions are confirmed by experiment for many commonly-used dielectrics (called *homogeneous, linear, isotropic dielectrics*), at least to an approximation which is adequate for most engineering applications. Therefore, write

$$(10.151) \qquad\qquad \mathbf{P} = \epsilon_0 \chi \mathbf{E}$$

in which $\epsilon_0 \chi$ is the factor of proportionality between $\mathbf{P}$ and $\mathbf{E}$ and χ is called the *dielectric susceptibility*. From Eq. (10.150)

$$(10.152) \qquad \mathbf{D} = \epsilon_0(1 + \chi)\mathbf{E} \quad \text{coulombs per square meter}$$

Then the *dielectric constant* ϵ/ϵ_0 is

$$(10.153) \qquad\qquad\qquad\qquad\qquad \text{(dimensionless)}$$
$$\epsilon_0$$

so that $\mathbf{D} = \epsilon \mathbf{E}$, which is Eq. (10.104). It follows from Eq. (10.148) that, *if ϵ/ϵ_0 is a constant*, then throughout all linear dielectrics, *except in surfaces* where $\mathbf{D}$ is discontinuous,

$$(10.154) \quad \int_v \mathbf{\nabla} \cdot \mathbf{D} \, dv = \int_v \rho \, dv \qquad \text{or} \qquad \cdot d\mathbf{S} = \int_v \rho \, dv$$

or, in differential form

$$(10.155) \qquad\qquad \mathbf{\nabla} \cdot \mathbf{D} = \rho$$

Equation (10.155) is one of Maxwell's equations (see footnote on page 265).

Note that it is the electrons of a dielectric, not the positive charges, that atomic theory postulates as the movable charges in a dielectric. The theory outlined above is not invalidated by this fact. The ρ_b described above is chosen as the positive charge density because the electrostatic-field intensity $\mathbf{E}$ is conventionally the force per unit positive charge.

If electrons in a dielectric do move when an electric field is applied, then it might be expected that an alternating field would produce an oscillating motion of the electrons; during such motion, losses would occur if there were any "frictional" forces opposing such motion.   Such losses are measurable; these effects are called *dielectric hysteresis*.   These phenomena are practically important at frequencies greater than a few hundred cycles per second, and sometimes at power frequencies (26 to 60 cycles per second).[8]

10.14. Lines or Tubes of Intensity and Flux Density; Equipotential Surfaces.—In a diagram representing a problem in electrostatics, it is often helpful to imagine lines everywhere tangent to the direction of the electric-field intensity **E** or to the (same) direction of the electric flux density **D**.   If the number of such lines through a small area perpendicular to the field is proportional to the magnitude of **E**, the lines become a pictorial representation of the field.   A set of surfaces everywhere perpendicular to the lines representing **E** can then represent *equipotential surfaces, i.e.,* surfaces over which the electrostatic potential is constant.   A network of lines can be drawn on a plane—one set representing **E** and the other set representing the intersection of equipotential surfaces with the plane.   An example is Fig. 10.04, involving a problem discussed in Sec. 10.09.

Vector fields are more clearly understood by some students in terms of this concept, suggested, and used with consummate good judgment, by Faraday.   He represented vector fields by *lines* or *tubes*.   Thus the field produced by an isolated small charge *Q* coulombs may be thought of as a point from which radial straight lines emanate into space, and the flux of **E** through a surface in the field is simply the number of lines through this surface.   The number of lines assumed to emanate from a charge is proportional to the charge, but the factor of proportionality is arbitrary.   The direction of the lines is everywhere the direction of the field vector.

Hypothetical lines of **D** begin on positive charges and end on negative charges.   At a boundary surface between two different dielectrics on which there are no free charges the lines representing **D** traverse the boundary unchanged, but there are more lines of **E** approaching one side of the surface than there are leaving the other side; the lines of **E** terminate at bound charges *or* at free charges.

There are vectors associated with the magnetic field that can be similarly represented by lines. Thus **H**, the magnetic-field intensity, is a vector whose magnitude changes at the boundary surface between the end of a permanent magnet and the surrounding air, while lines representing magnetic flux density **B** are continuous *closed* lines. These concepts are discussed in subsequent chapters.

Lines of flux density **D** incident upon a surface d**S** at an angle θ constitute a flux $d\psi$ through the surface equal in magnitude to $D \cos \theta \, dS = \mathbf{D} \cdot \mathbf{dS}$.

The student should sketch the hypothetical lines that may be associated with the problems discussed in this chapter, as indicated, for example, in Sec. 10.09 (Fig. 10.04). Problems involving electric and magnetic fields, illustrated by clear, accurate drawings that include lines representing vector fields and equipotential surfaces, are discussed in detail in one of the references.[9]

References

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3. *A Very Accurate Test of Coulomb's Law of Force between Charges;* S. J. Plimpton and W. E. Lawton; *Phys. Rev.,* **50**, 1066–1071, 1936.
4. *Treatise on Electricity and Magnetism;* J. H. Jeans; 4th ed., pp. 185–299, Cambridge University Press, London, 1923.
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Problems

10.01. A square plate having sides a cm. long is charged uniformly with a surface-charge density σ coulombs per cm². Calculate the electric-field intensity at a point $a/2$ cm. from the plate on the line that is perpendicular to the plate at its center.

10.02. A conductor A acquires a charge Q. When this charged conductor A is brought into contact with a second insulated conductor B, which is initially uncharged, a charge $Q/8$ is transferred from A to B. If A is repeat-

edly charged and then brought into contact with B, what is the maximum charge that can be acquired by B?

10.03. Calculate the potential and the electric-field intensity at any point outside: (*a*) a uniformly charged sphere of radius R and total charge Q; (*b*) a sphere of radius R whose *surface* is uniformly charged, total charge Q; (*c*) a point charge Q; (*d*) an infinite cylinder with uniform volume-charge density, radius R and charge per unit length Q; (*e*) an infinite line charge of Q coulombs per unit length. Calculate the potential and field *within* the sphere of *a*, of *b*, and the cylinder of *d*.

10.04. A charge Q is placed at a distance d from an infinite conducting plane whose potential is taken as zero. Prove that the charge on any area of the plane is proportional to the solid angle subtended by the area at the point where Q is located.

10.05. An infinite circular cylinder of radius R and dielectric constant k_2 is placed in a region of dielectric constant k_1 in which there was a uniform field E_0 perpendicular to the line that subsequently coincides with the axis of the cylinder. Calculate the potential inside the cylinder and the potential outside the cylinder.

10.06. If the field at the earth's surface is 300 volts per meter downward, and the field at a distance of 1,400 m. above the earth's surface is 20 volts per m. downward, calculate: (*a*) the mean volume-charge density in the region from the earth's surface to a height of 1,400 m.; and (*b*) the total charge on the surface of the earth if these data were valid for all points on the earth's surface.

10.07. A cylinder 20 cm. long and 8 cm. in diameter carries a surface-charge density 2×10^{-16} coulomb per cm². Calculate the potential and the electric field at a point on the extended axis of the cylinder, 10 cm. from the nearer end face.

10.08. Charges $+4Q$ and $-Q$ are placed at points A and B. Find the point C at which the electrostatic-field intensity is zero and show that the line of force that passes through C intersects AB at A at an angle of 60 deg. and intersects BC at an angle of 90 deg.

10.09. Prove that if two charged concentric spherical conductors are connected, the charge on the inner conductor becomes zero. Prove that if the law of force were $F \propto 1/r^{2+\alpha}$, then a charge Q' proportional to α and to the charge Q on the outer sphere would remain on the inner sphere.

10.10. The dielectric strength of air is approximately 30,000 volts per cm. Calculate the maximum charge that can be placed on an isolated sphere 12.5 cm. in diameter in air. What is the potential of the sphere when it is charged in this manner?

10.11. Two concentric cylindrical conductors are insulated by air. The inner radius of the outer conductor is 5 cm.; the radius of the inner conductor is r cm. Choose r so that the two conductors can be operated with maximum potential difference between them and calculate the potential difference. What effect is produced by surface irregularities in the conductors?

10.12. If two spheres of radius r cm. are charged with $+Q$ and $-Q$ and their centers are d cm. apart, calculate the difference of potential between them at which a spark will occur. Check for $2r = 25$ cm. and $d = 3.5$ cm. against the A.I.E.E. data for sphere-gap measurements.

10.13. Design superficially a cable having two concentric cylindrical conductors separated by two layers of insulation having dielectric constants 3 and 6 and dielectric strengths 70×10^6 volts per meter and 35×10^6 volts per meter respectively. The cable is to operate at 66,000 volts rms, the inner conductor has a cross-sectional area of 2,000,000 circular mils, and the volume of the insulation is to be as small as possible.

10.14. A charge Q is placed d m. from the center of a grounded conducting sphere of radius R. Prove that the ratio of the surface charge on the part of the sphere that is visible from Q is to the surface charge that is invisible as $(d + R)^{\frac{1}{2}}$ is to $(d - R)^{\frac{1}{2}}$.

10.15. Show that Laplace's equation in two dimensions is satisfied by a function

$$\sum_{n=-\infty}^{n=\infty} r^n(\sin n\theta - \cos n\theta)$$

in which n is always an integer and r and θ are polar coördinates.

10.16. If a conducting sphere of radius R and potential zero is placed in a region in which there is a uniform field produced by charges far from the site of the sphere, calculate the potential at any point outside the sphere but near to its surface, the surface-charge density at any point on the sphere, and the point or points where the electric field is zero.

10.17. The tension in a soap bubble of radius R is T and the atmospheric pressure is P. The bubble is touched by a wire connected to a large conductor of potential V. Calculate r, the radius of the bubble after it is touched by the wire, assuming the soap film to be a conductor.

10.18. Calculate the field produced by a charge Q placed with respect to two perpendicular planes of potential zero as shown below.
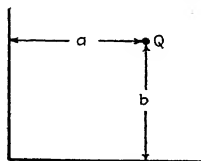
PROB. 10.18.

Calculate the distribution of charge on the planes.

10.19. Repeat Prob. 10.18 for the arrangement shown below.
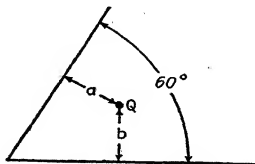
PROB. 10.19.

10.20. What is the least positive charge that must be given to an insulated spherical conductor in order that the surface density be everywhere positive, if there is a charge Q at a distance d from the center of the sphere?

10.21. Two regions of materials having dielectric constants k_1 and k_2 are separated by a plane. A charge Q_1 is placed in the material of dielectric constant k_1, a m. from the plane, and a charge Q_2 is placed in the material of dielectric constant k_2, b m. from the plane; Q_1, Q_2, and the point of intersection of a and b and the plane are collinear. Sketch the field of **E** and the field of **D** and calculate the surface polarization on each material at an arbitrary point on the plane distant $(a^2 + x^2)^{\frac{1}{2}}$ meters from Q_1.

CHAPTER XI

ELECTROSTATIC CAPACITANCE

11.01. Capacitance and Electrostatic Induction Coefficients.— The condenser as a circuit element was discussed in Chaps. I to VIII. It is the purpose of this chapter to derive methods for calculating the capacitances of condensers of various forms and to analyze the mechanical forces and the potential energy related to a few configurations of conductors in electrostatic fields.

Consider a system of charged conductors in none of which a conduction current exists; assume that they are separated by "perfect" dielectrics on no part of the surface of which a free charge exists; and let all the conductors and all the dielectrics in the field be fixed relative to one another and remain unaltered in shape and dimensions. By a "perfect" dielectric is here meant one in which there are no free charges and therefore in which no conduction current can be established. These ideal conditions can never be absolutely realized, but they can be so nearly approached that in the analysis of most electrostatic phenomena no appreciable error is made by assuming them to be valid. Throughout this chapter all the conditions here stated will be assumed to exist; only macroscopic potentials and intensities will be considered, but for brevity the descriptive adjective macroscopic will be omitted; and also for brevity electrostatic potential and electrostatic intensity will be referred to simply as electric potential and electric intensity respectively.

In a system of the type just described each conductor is at the same electric potential throughout. In particular the surface of each conductor is an *equipotential surface, i.e.,* a surface every point of which is at the same electrostatic potential. Let there be $(n + 1)$ conductors in the system under consideration and designate these conductors as No. 0, No. 1, No. 2, . . . No. n. When these conductors are so far removed from all other conductors, or are so *shielded* from these other conductors, that charges on the latter affect in no way the amount or distribution

of charges on the specified $(n + 1)$ conductors, and the algebraic sum of all the charges on these $(n + 1)$ conductors is zero, then the specified system of $(n + 1)$ conductors is said to be an *electrostatically independent system.*

Designate by Q_0, Q_1, Q_2, . . . Q_n the total net charges on the respective conductors. Then in an electrostatically independent system

(11.01) $Q_n)$ coulombs

choose any one of the $(n + 1)$ conductors as a *conductor of reference* and designate this reference conductor as conductor No. 0. It is frequently convenient to choose the earth as the reference conductor, but such a choice is in no way imperative. From the expression for the electrostatic potential at any point due to the free and bound charges at the contact surfaces between any number of different media (conductors or dielectrics), Eq. (10.093) of Sec. 10.11, it may readily be shown that the *differences of potential* between the several conductors and the conductor of reference may always be expressed in the form

$$V_{20} = A_{21}Q_1 \quad \begin{matrix} + A_{12}Q_2 \\ \end{matrix} \quad \begin{matrix} + A_{1n}Q_n \\ + A_{2n}Q_n \end{matrix}$$

(11.02)   volts

$$V_{n0} = \qquad + A_{n2}Q_2 +$$

where the A's are all constants* which depend upon (a) the shape and dimensions of each conductor, (b) the shape and dimensions of each dielectric, (c) the relative positions of these several media, and (d) the permittivity of each dielectric in the system. These A coefficients are called *potential coefficients.*

Consider first the case of an electrostatically independent system of two conductors. Choose as the reference conductor that one of these conductors which is negatively charged, and let Q_1 be the charge (positive by hypothesis) on the other conductor. Then Eqs. (11.01) and (11.02) become

(11.03) $Q_0 = -Q_1$
(11.04) $V_{10} = A_{11}Q_1$

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* Provided the permittivity of each dielectric in the system is constant.

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by the symbol C_{11}, or $\sim$ C without subscripts,
called the *capacitance* of this capacitor, *viz.*,

$$(11.05) \qquad C = \frac{Q_1}{V_{10}} = \frac{1}{A_{11}}$$

The unit of capacitance in the mks system is called the *farad*.
The value of the capacitance in any system of units depends
upon the same factors as determine the value of the potential
coefficient A_{11}, which factors are listed under a, b, c, and d in the
preceding paragraph.

The formulas for the capacitance of several simple forms of
capacitors will be deduced later. However, before developing
these formulas, it is of interest to extend the concept of capaci-
tance to the case of more than two conductors. Consider, for
example, an electrostatically independent system of three con-
ductors and choose any one of these as the reference conductor.
Then Eqs. (11.01) and (11.02) become

$$(11.06) \qquad Q_0 = -(Q_1 + \text{(}$$
$$(11.07) \qquad V_{10} = A_{11}Q_1 + .$$
$$(11.08) \qquad V_{20} = A_{21}Q_1 +$$

The solution of Eqs. (11.07) and (11.08) for Q_1 and Q_2 is

$$(11.09) \qquad Q_1 = C_{11}V_{10} +$$
$$(11.10) \qquad Q_2 = C_{21}V_{10} +$$

where

$$A_2$$

$$(11.11) \qquad \frac{}{A_{11}A_{22} - A_{12}A_{21}}$$
$$\frac{-A_{12}}{A_{11}A_{22} - A_{12}A_{21}}$$
$$\frac{-A_{21}}{A_{11}A_{22}}$$

The C coefficients in Eqs. (11.09) to (11.11) all have the
dimensions of capacitance, and in the mks system of units they
are expressed in farads. They are sometimes referred to as
coefficients of electrostatic induction. It will be shown subsequently

that the potential coefficient A_{12} is equal to the potential coefficient A_{21}, so that for a three-conductor system there are only three independent capacitances, namely C_{11}, C_{22}, and $C_{12} = C_{21}$.   More generally it can be shown that in an electrostatically independent system of $(n + 1)$ conductors

(11.12) $$A_{hk} = A_{kh}$$

and therefore

(11.13) $$C_{hk} = C_{kh}$$

Note that each of the capacitances in Eq. (11.11) has a simple physical significance.   For example, C_{11} is the capacitance of the capacitor formed by conductor No. 1 and conductors Nos. 2 and 0, when the latter are connected by a metallic wire of infinitesimal cross section ($V_{20} = 0$).   Similarly, C_{22} is the capacitance of the capacitor formed by conductor No. 2, insulated from conductors Nos. 1 and 0, when the latter are connected by a metallic wire of infinitesimal cross section.   The coefficient $C_{12} = C_{21}$ is equal to the negative of the ratio of the charge on conductor No. 1 to the voltage drop from conductor No. 2 to conductor No. 0 when No. 1 is insulated from No. 2 and No. 1 is connected to No. 0 by a wire of infinitesimal cross section ($V_{10} = 0$).

Experiment shows that energy is always required to charge a capacitor, or any system of conductors, and that this energy may be considered as *stored* in the electrostatic field in the surrounding dielectrics.   Consider first the case of a two-conductor system that forms a simple capacitor and designate by q_1 the charge on the positive plate at any instant of time.   From Eq. (11.04) the potential drop from conductor No. 1 to conductor No. 0 at this instant is*

(11.14) $$v_{10} = A_{11}q_1$$

Imagine an infinitesimal positive charge dq_1 to move from conductor No. 0 to conductor No. 1.   The work required to move this charge against the electrostatic intensity due to the absolute potential that corresponds to the two charges q_1 and $-q_1$ on conductors Nos. 1 and 0 respectively is then

(11.15) $$(v_1 - v_0)\, dq_1 = v_{10}\, dq_1 = A_{11}q_1\, dq_1$$

* Do not confuse the lower-case v here used for potential drop with the use of the same symbol in Chap. X for volume.

To build up the charge on conductor No. 1 from zero to the value Q_1 by moving successive elementary charges dq_1 from No. 0 to No. 1 therefore requires an amount of work

$$(11.16) \qquad W = \int_{q_1=0}^{q_1=Q_1} A_{11}q_1 \, dq_1 = \tfrac{1}{2}A_{11}Q_1^2 \qquad \text{joules}$$

By definition of energy, this work is the amount of energy required to charge the capacitor.

Consider next a three-conductor system initially uncharged. Imagine a charge dq_1 to move from No. 0 to No. 1, but no charge to move from No. 0 to No. 2 or from No. 1 to No. 2. As in the case of the two-conductor system, the work required under these conditions to build up the charge on conductor No. 1 to the value Q_1 is

$$(11.17) \qquad\qquad W_1 = \tfrac{1}{2}A_{11}Q_1^2 \qquad\qquad \text{joules}$$

Now hold the charge constant on conductor No. 1 at the value Q_1 and consider any instant of time when the total charge that has been moved from conductor No. 0 to conductor No. 2 is q_2. Under these conditions the net charge on conductor No. 0 is

$$(11.18) \qquad\qquad q_0 = -(Q_1 + q_2)$$

and the voltage drop from conductor No. 2 to conductor No. 0 is, from Eq. (11.08)

$$(11.19) \qquad\qquad v_{20} =$$

Now imagine an infinitesimal charge dq_2 to move from conductor No. 0 to conductor No. 2. The work required to move this charge against the electrostatic intensity due to the absolute potential that corresponds to the three charges q_0, Q_1, and q_2 is then

$$(11.20) \quad (v_2 - v_0) \, dq_2 = v_{20} \, dq_2 = A_{21}Q_1 \, dq_2 + A_{22}q_2 \, dq_2.$$

To build up the net charge on conductor No. 2, in the manner just specified, from zero to the value Q_2, by moving successive infinitesimal charges dq_2 from No. 0 to No. 2 therefore requires an amount of work

$$(11.21) \qquad\qquad W_2 = A_{21}Q_1Q_2 + \tfrac{1}{2}A_{22}Q_2^2 \qquad\qquad \text{joules}$$

The sum of Eqs. (11.20) and (11.21) then gives the total energy

required to charge the three conductors, which total is

$$(11.22)* \qquad W = \tfrac{1}{2}A_{11}Q_1^2 + A_{21}Q_1Q_2 + \tfrac{1}{2}A_{22}Q_2^2 \qquad \text{joules}$$

By the same argument, if conductor No. 2 had been charged first and then conductor No. 1 charged, with Q_2 held constant, the total energy required to charge this system would be

$$(11.23) \qquad W = \qquad Q_1^2 + A_{12}Q \qquad \tfrac{1}{2}A_{22}Q_2^2 \qquad \text{joules}$$

From the principle of conservation of energy the two expressions Eqs. (11.22) and (11.23) must be identical. Hence the potential coefficients A_{12} and A_{21} must have identically the same values. Also note that from Eqs. (11.07) and (11.08). and the relation that $A_{21} = A_{12}$, either of the two expressions Eqs. (11.22) and (11.23) for the total energy required to charge the system of three conductors may be written

$$(11.24) \qquad W = \tfrac{1}{2}Q_1V_{10} + \tfrac{1}{2}Q_2V_{20} \qquad \text{joules}$$

By a similar analysis it can be shown that for any electrostatically independent system of $(n + 1)$ conductors, with conductor No. 0 chosen as the reference conductor,

$$(11.25) \qquad A_{hk} = A_{kh}$$

and the total energy stored in the electrostatic field is

$$(11.26) \qquad W = \cdot \qquad \text{joules}$$

11.02. Spherical Condenser.—It is the purpose of this section to derive from Eq. (11.05) the formula for the capacitance of two concentric spherical conductors of radii R_1 and R_2 ($R_2 > R_1$) separated by a medium of dielectric constant 1, and then to calculate the potential and induction coefficients for the system. If the charges on the spheres are Q and $-Q$, the charges become uniformly distributed over the surfaces of the spheres. The lines of **E** are there-
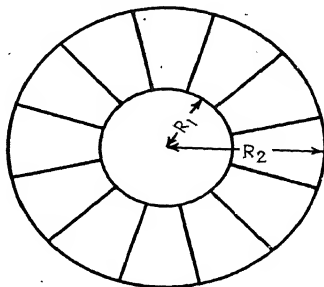
FIG. 11.01.

* See Problem 11.28.

fore radial, as shown in cross section Fig. 11.01. This field is precisely the field that would be produced on the surfaces representing the spheres, if they were removed and a single point charge Q were placed at their common center. Therefore

$$(11.27) \qquad V_1 - V_2 = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad \text{volts}$$

$$(11.28) \qquad C = \frac{Q}{V_1 - V_2} = \frac{4\pi\epsilon_0 R_1 R_2}{R_2 - R_1} \qquad \text{farads}$$

Note that if $R_2 >> R_1$ Eq. (11.28) reduces to

$$(11.29) \qquad C = 4\pi\epsilon_0 R_1 \qquad \text{farads}$$

This is sometimes described as the "capacitance" of an isolated sphere. It is, as this derivation shows, the capacitance between two concentric spheres, one of which is very much larger than the other.

It was shown in Chap. IV that the resulting capacitance C of several condensers C_1, C_2, . . . C_n, connected in parallel can be calculated from

$$(11.30) \qquad C = C_1 + C_2 + \cdot \qquad + C_n \qquad \text{farads}$$

It is common practice to refer to the earth as a body having a reference "zero" potential, as noted in Sec. 11.01. Thus a conductor having no current in it can be maintained at this zero potential by connecting it to ground by means of a wire. Suppose that a concentric spherical condenser is far enough from the surface of the earth so that the outer sphere is, for practical purposes, an "isolated" sphere. If the outer sphere is grounded, the capacitance of the condenser is

$$(11.28) \qquad \frac{4\pi\epsilon_0 R_1 R_2}{R_2 - R_1} \qquad \text{farads}$$

On the other hand, if the inner sphere is grounded, the capacitance of the condenser consists of two parts, (1) the capacitance from the outer sphere to ground in parallel with (2) the capacitance from the outer sphere to the inner sphere:

$$(11.31) \qquad C' = 4\pi\epsilon_0\left(R_2 + \frac{R_1 R_2}{R_2 - R_1}\right) = \frac{4\pi\epsilon_0 R_2^2}{R_2 - R_1} \qquad \text{farads}$$

Thus care must be taken to know just what is meant by the word

capacitance in any problem in which more than two conductors are involved.

By using the relation of Eq. (11.31) the coefficients of potential and induction can be calculated.   The equations involving them are Eqs. (11.07) to (11.10)

$$V_{10} = A_{11}Q_1 + A_{12}Q_2 \qquad Q_1 = C_{11}V_{10} + C_{12}V_{20}$$
$$V_{20} = A_{21}Q_1 + A_{22}Q_2 \qquad Q_2 = C_{12}V_{10} + C_{22}V_{20}$$

By assigning particular values to Q_1 and Q_2, the coefficients A_{11}, $A_{12} = A_{21}$, A_{22} can be calculated, using the relation of Eq. (11.20):

(11.32)
$$Q_2 = 0 \qquad \qquad \frac{V_1}{} \quad \frac{1}{4\pi\epsilon_0 R_2}$$

$$Q_1 = 0 \qquad A_{22} = \frac{V_2}{Q_2} = \frac{1}{}$$

and when $Q_1 = 0$, $V_1 = \qquad$ regardless of the value of Q_2 so that

(11.33)
$$_{12} = \frac{1}{4\pi\epsilon_0 R_2}$$

The coefficients C_{11}, C_{12}, C_{22}, calculated from the values of the A's listed above, are

(11.34)
$$C_{11} = 4\pi\epsilon_0 \cdot \frac{R_1 R_2}{}$$

$$= -4\pi\epsilon_0 \quad \frac{R_1 R_2}{}$$

$$C_{22} = \qquad R_2^2$$

The "capacitance" of the concentric spherical condenser for two cases, the outer sphere grounded ($V_2 = 0$) and the inner sphere grounded, ($V_1 = 0$) are respectively

(11.35)
$$C = 4\pi\epsilon_0 \; \frac{}{R_2 - }$$

(11.36)
$$C' = \quad \epsilon_0 \; \frac{}{R_2 - }$$

corresponding to the derivations of Eqs. (11.28) and (11.31) given above.

Problem: Calculate the "capacitance" of a concentric spherical condenser with inner sphere grounded when the dielectric constant of the medium

between the spheres is ϵ/ϵ_0 and the dielectric constant of the space surrounding the larger sphere is 1.

11.03. Parallel-plate Condenser.—The approximate capacitance of a condenser consisting of two parallel conducting plates of area S is derived in this section. The exact capacitance can be calculated only by taking into account the *edge effects, i.e.,* the effects of phenomena in the regions near the peripheries of the plates where the electrostatic flux density $\mathbf{D}$ is not perpendicular to the plates. The edge effects are small if the distance of separation of the plates s is very much smaller than the linear dimensions of each plate ($\sqrt{S}$). Since this is the condition often encountered in practice, the approximation given below is practically useful.
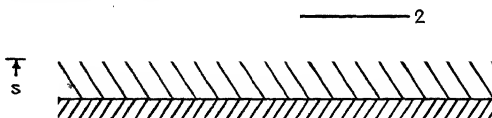
FIG. 11.02.

A cross section of the parallel plate condenser is shown in Fig. 11.02. Two dielectrics of thicknesses $(s - t)$ and t are shown. It is assumed that the plates connected to the leads 1 and 2 are equipotential surfaces, and that $\mathbf{D}$ and $\mathbf{E}$ are everywhere perpendicular to the plates. According to Eq. (10.040), if a charge Q is placed on plate 1 and a charge $-Q$ on plate 2, the electrostatic flux density $\mathbf{D}$ is directed vertically upward in Fig. 11.02 and its magnitude is everywhere.

$$(11.37) \qquad\qquad = \frac{Q}{S} \quad \text{coulombs per square meter}$$

The magnitude of E in the material of dielectric constant ϵ_1/ϵ_0 differs from the magnitude of E in the second dielectric of dielectric constant ϵ_2/ϵ_0

$$(11.38) \qquad D = \epsilon_1 E_1 = \epsilon_2 E_2 \quad \text{coulombs per square meter}$$

Finally, the difference in potential V between the two plates is the line integral of $\mathbf{E}$ from plate 1 to plate 2, which in this special case is

$$(11.39) \qquad\qquad V = E_1 t + E_2 (s - t) \qquad\qquad \text{volts}$$

By appropriate juggling of Eqs. (11.37) to (11.39), and Eq. (11.05) the capacitance turns out to be

$$(11.40) \qquad C = \frac{}{\overline{\quad + \epsilon_1(s - t)}} \qquad \text{farads}$$

If the dielectric throughout the space between the plates is a vacuum, Eq. (11.40) reduces to

$$(11.41) \qquad C_1 = \epsilon_0 \frac{S}{s} \qquad \text{farads}$$

which holds approximately for a condenser having air at normal temperature and pressure as a dielectric. If a slab of dielectric of constant ϵ_1/ϵ_0 and thickness t is inserted in the air condenser, Eq. (11.41), the capacitance increases to

$$(11.42) \qquad C_2 = \qquad \text{farads}$$

and the increase in capacitance is

$$(11.43) \qquad C_2 - C_1 = \frac{\epsilon_0}{} \qquad \frac{\epsilon_0)t}{(s - t)} \Bigg] \qquad \text{farads}$$

11.04. The Capacitance between Two Parallel Cylindrical Conductors.[1,2]—The capacitance per unit length of two long
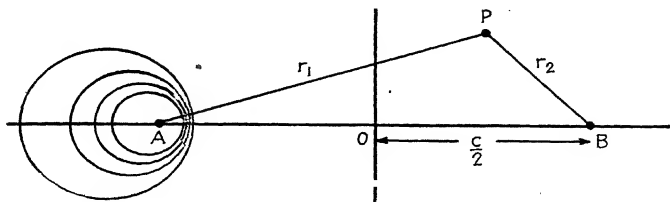
FIG. 11.03.—Line charges: $+Q$ coulombs per meter along A; $-Q$ coulombs per meter along B.

parallel cylindrical wires such as the wires of a transmission line can be conveniently calculated by first analyzing the scalar field of the electrostatic potential around two hypothetical *line charges.* Figure 11.03 shows the traces on the paper at A and B of two lines along which charge is assumed to be uniformly distributed (Q coulombs per meter). If the line is sufficiently long the field **E** is everywhere perpendicular to the wires except near the ends of the line. Thus a map of **E** and V for the plane

of the paper Fig. 11.03 is a solution for any point along the line except for those points near the ends.

First consider the line charge A alone.   The field $\mathbf{E}_1$ is radial, being directed outward from A.   Its magnitude at a point such as P is, from Gauss's law,

$$(11.44) \qquad \int_S \mathbf{E}_1 \cdot \mathbf{dS} = 2\pi r_1 E_1 = \frac{Q}{\epsilon_0}$$

$$E_1 = \frac{Q}{2\pi r_1 \epsilon_0} \qquad \text{volts per meter}$$

and the potential at point P is approximately (for $R >> r_1$)

$$(11.45)^* \quad V_1 = -\int_R^{r_1} \frac{Q}{2\pi r_1 \epsilon_0}\, dr_1 = -\frac{Q}{2\pi \epsilon_0} (\log r_1 - \log R) \quad \text{volts}$$

Similarly the potential at P, produced by $-Q$ along B, is

$$(11.46) \qquad V_2 = \frac{Q}{2\pi \epsilon_0} (\log r_2 - \log R) \qquad \text{volts}$$

Therefore the potential V at a point such as P is the sum of V_1 and V_2

$$(11.47) \qquad V = \frac{Q}{2\pi \epsilon_0} \log \frac{r_2}{r_1} \qquad \text{volts}$$

It follows that the equipotential surfaces of the scalar field of V are those surfaces perpendicular to the plane of Fig. 11.03 represented by lines on the figure, for which

$$(11.48) \qquad \frac{r_2}{r_1} = K \qquad \text{a constant}$$

Therefore, writing Eq. (11.48) in terms of cartesian coordinates with O as origin, the equation of a line in Fig. (11.03) representing an equipotential surface is

$$(11.49) \quad \frac{\left(\dfrac{c}{2} - x\right)^2 + y^2}{\left(\dfrac{c}{2} + x\right)^2 + y^2} = K^2 \qquad \text{or}$$

$$\left(x + \frac{K^2 + 1}{K^2 - 1}\frac{c}{2}\right)^2 + y^2 = \frac{K^2 c^2}{(K^2 - 1)^2}$$

* As a practice problem calculate V_1 directly and derive $\mathbf{E}_1$ from it.

This is the equation of a family of circles of radius $Kc/|K^2 - 1|$ with centers at x_0, y_0

(11.50) $\qquad \sim \; - \; \dfrac{K^2 + 1}{K^2 - 1}\dfrac{c}{2} \qquad \sim \; - \; \cap$

The equipotential surfaces are cylinders with axes perpendicular to the plane of the circles given by Eq. (11.49). Note that the product of the distances from x_0, y_0 to A and B is identically equal to the square of the radius of the circle, for all values of K

(11.51) $\qquad \left(x_0 - \dfrac{c}{2}\right)\left(x_0 + \dfrac{c}{2}\right) = \dfrac{K^2 c^2}{(1 - K^2)^2}$

The points A and B, Fig. 11.03, are called the *inverse points* of the circles of radius $Kc/|K^2 - 1|$ about the points $+x_0$, $-x_0$ (y being zero for all four points).
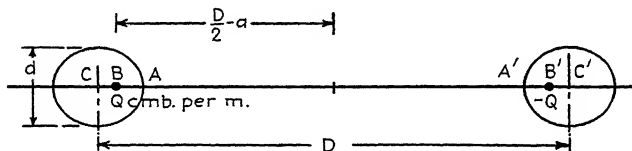
FIG. 11.04.

The important conclusion is therefore that two cylinders of radius $Kc/|K^2 - 1|$ placed with their centers at $+x_0$ and $-x_0$ are equipotential surfaces in the fields of two line charges placed at $x = c/2$ and $x = -(c/2)$ in the manner indicated in Fig. 11.03. The surfaces of cylindrical conductors *are* equipotential surfaces in an electrostatic field. Their capacitance per unit length can therefore be calculated in the manner described below.

Figure 11.04 shows the cross sections of two cylindrical conductors of diameter d with their centers separated by a distance D. The points B, B' are inverse points of the circles drawn about the centers C and C'. If charges Q and $-Q$ coulombs per meter are distributed along B and B', then the cylinders represented by the two circles are equipotential surfaces. Define a quantity a as follows:

(11.52) $\qquad CB = C'B' = a$

and note from Eq. (11.51) that

(11.53) $\qquad a(D - a) = \dfrac{d^2}{4}$

so that

(11.54)
$$a = \frac{D \pm \sqrt{D^2 - d^2}}{2}$$

The value of a having a negative sign in front of the square root is relevant to this problem.   Now calculate the potentials of points A and A' from Eq. (11.47)

(11.55)
$$V_A = \frac{Q}{2\pi\epsilon_0} \log \frac{D - \frac{d}{2} - a}{\frac{d}{2} - a}$$

(11.56)
$$V_{A'} = \frac{Q}{2\pi\epsilon_0} \log \frac{\frac{d}{2} - a}{D - \frac{d}{2} - a}$$

Then

(11.57)
$$V_A - V_{A'} = \frac{Q}{\pi\epsilon_0} \log \frac{D - \frac{d}{2} - a}{\frac{d}{2} - a}$$

which becomes, after the value of a is substituted from Eq. (11.54),

(11.58)
$$V_A - V_{A'} = \frac{Q}{\pi\epsilon_0} \log \left[\frac{D}{d} + \sqrt{\left(\frac{D}{d}\right)^2 - 1} \right]$$

Therefore the capacitance per unit length of two long parallel cylindrical conductors in air, far from other conductors, is

(11.59)
$$C = \frac{\pi\epsilon_0}{\log \left[\frac{D}{d} + \sqrt{\left(\frac{D}{d}\right)^2 - 1} \right]} = \frac{\pi\epsilon_0}{\cosh^{-1} \frac{D}{d}}$$

farads per meter

If $D > 10d$ the following approximation is accurate to 0.1 per cent or less

(11.60)
$$C = \frac{\pi\epsilon_0}{\log \frac{2D}{d}} \qquad (D > 10d) \text{ farads per meter}$$

The distribution of charge on the parallel conductors is analyzed in one[2] of the references listed at the end of this chapter.

11.05. A Method of Measuring Capacitance Indirectly.—It often happens that the capacitance of two conductors cannot be calculated conveniently by the analytical methods of which examples have been discussed in this chapter. In such cases the conductors can often be immersed in a liquid of known conductivity γ, and the conductance G of the system can be measured. The capacitance can then be calculated in the manner described below.

If the two conductors are equipotential surfaces when they are immersed in the conducting liquid with a current flowing between them, the conductance G, the measured current I, and the measured potential drop between them are related

$$(11.61) \qquad G = \frac{I}{V} = \frac{\int_S \mathbf{J} \cdot \mathbf{dS}}{\int_1^2 \mathbf{E} \cdot \mathbf{ds}} \qquad \text{mhos}$$

This follows from the facts that the total current I is the integral of the current density $\mathbf{J}$ over the surface S of either conductor and that the difference of potential between the conductors 1 and 2 is equal to the line integral of $\mathbf{E}$ along any path s from one conductor to the other. However, in a conductor, the potential gradient $\mathbf{E}$ is determined at a point by the ratio of the current density $\mathbf{J}$ to the conductivity γ, *i.e.*, $\mathbf{J} = \gamma\mathbf{E}$. Therefore

$$(11.62) \qquad G = \frac{\gamma \int_S \mathbf{E} \cdot \mathbf{dS}}{\int_1^2 \mathbf{E} \cdot \mathbf{ds}} \qquad \text{mhos}$$

The capacitance C between two conductors separated by a medium of permittivity ϵ is

$$(11.63) \qquad C = \frac{Q}{V} = \frac{\int_S \mathbf{D} \cdot \mathbf{dS}}{\int_1^2 \mathbf{E} \cdot \mathbf{ds}} = \frac{\epsilon \int_S \mathbf{E} \cdot \mathbf{dS}}{\int_1^2 \mathbf{E} \cdot \mathbf{ds}} \qquad \text{farads}$$

Since the capacitance C and the conductance G are assumed to be constant throughout the ranges of $\mathbf{E}$ normally obtained either in conductors or in insulators, the value of $\mathbf{E}$ may be assumed to be the same for Eqs. (11.62) and (11.63).* From the ratio of

* The essential points implied here are that, in the conducting system, Eq. (11.62), and in the condenser, Eq. (11.63); (1) the stream lines of current

the two, the value of the capacitance is

(11.64) $$C = \frac{\epsilon G}{\nu}$$ farads

11.06. Mechanical Forces in Electrostatic Fields.—In general the potential energy of an electrostatic field is a function of the coordinates of points at which there is electric charge. If a charged conductor in such a field is assumed to move an infinitesimal distance ds, the mechanical force on the conductor can be calculated in terms of the change in potential energy thus
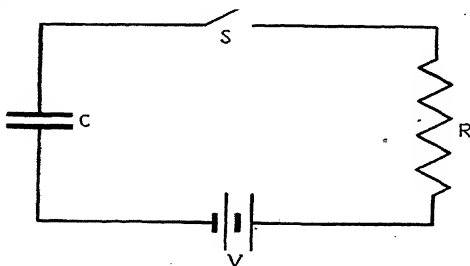
FIG. 11.05.

produced and the distance ds. The general problems are often difficult, but a few simple examples are presented below.

First consider a parallel plate condenser of capacitance C charged by means of a battery of voltage V and then disconnected from the battery by opening the switch S, Fig. 11.05. The condenser is charged to the voltage V so that its potential energy W is [see Chap. IV and Eq. (11.16)].

(11.65) $$W = \frac{1}{2}\frac{Q^2}{C} = \frac{Q^2}{2\epsilon_0 S} s$$ joules

If the distance of separation is increased slightly, the energy increases also although the charge remains constant. Therefore there is a force of attraction F on the plates equal to

(11.66) $$F = -\frac{dW}{ds} = \frac{-Q^2}{2\epsilon_0 S}$$ newtons

and the lines of the electric field **E** have the same shape and distribution; and (2) the equipotential surfaces are identical in shape and distribution.

Note that this method of calculating force is valid *only if Q is constant during the time of the assumed displacement ds.* Thus if the battery remains connected, and the plates are assumed to be separated by an amount ds beyond the original separation s, the calculation $-dW/ds$ is not equal to the mechanical force because the charge Q changes during the displacement ds.

The force between two parallel cylindrical conductors, with *fixed* charges Q and $-Q$ coulombs per meter, can be calculated in terms of this charge Q, the capacitance per unit length C, and the geometry of the system. It is clear from the symmetry of Fig. 11.04 that the force is a force of attraction along the lines joining the centers of the wires. Thus the force is

$$(11.67)* \quad F = -\frac{dW}{d(D)} = -\frac{Q^2}{2}\frac{d}{d(D)}\left[\frac{\cosh^{-1}(D/d)}{\pi\epsilon_0}\right]$$

$$F = -\frac{\pi\epsilon_0 V^2}{2\left[\cosh^{-1}(D/d)\right]^2\sqrt{D^2 - d^2}}\text{ newtons per meter}$$

Note from Eq. (11.65) and the relation $Q = DS$ (D is electric-flux density) for a parallel-plate condenser, the result

$$(11.68) \qquad W = \frac{1}{2}\frac{D^2}{\epsilon}Ss \qquad \text{joules}$$

so that the *energy per unit volume* is

$$(11.69) \qquad W' = \frac{1}{2}\frac{D^2}{\epsilon} = \frac{1}{2}\epsilon E^2 = \frac{1}{2}ED \quad \text{joules per cubic meter}$$

This concept has useful generalizations, which are discussed in Chap. XVI.

References

1. *A Treatise on the Theory of Alternating Currents;* Alexander Russell; pp. 99–102, Cambridge University Press, London, 1904.
2. *The Capacity between Two Equal Parallel Wires;* Harold Pender and H. S. Osborne; *Elec. World,* **56,** 667–670, Sept. 22, 1910.

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Principles of Electricity and Electromagnetism; G. P. Harnwell; pp. 1–72, McGraw-Hill Book Company, Inc., New York, 1938.
The Mathematical Theory of Electricity and Magnetism; J. H. Jeans; 5th ed., pp. 24–186, Cambridge University Press, London, 1927.

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* The minus signs in Eqs. (11.66) and (11.67) signify that the forces are in the direction of *decreasing* separation of the conductors.

Principles of Electricity; L. Page and M. I. Adams, Jr.; pp. 1–81, D. Van Nostrand Company, Inc., New York, 1931.

Electricity and Magnetism; S. G. Starling; 6th ed., pp. 105–169, Longmans, Green and Company, New York, 1937.

Problems

11.01. The drawing below represents the cross section through the centers of a set of coaxial circular pieces; B and C are metal disks, D is a glass disk (dielectric constant $\epsilon/\epsilon_0 = 6$), and A is an annular metal ring.
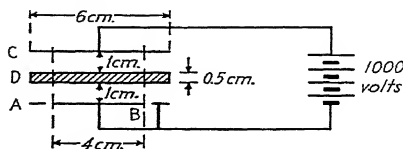
PROB. 11.01.

If the spaces between C and AB, outside of the glass disk, contain air, calculate: (*a*) the surface charge on B and its distribution; (*b*) the surface charge on C and its distribution, within the 4-cm. circle indicated by the dotted lines; (*c*) the normal polarizations on the two surfaces of the glass disk, within the central 4-cm. circles; (*d*) **E** and **D** along the axis, at every point from B to C; (*e*) the total electrostatic flux from B; (*f*) the ratio of the surface charge on B to the potential difference between B and C. What is the purpose of the ring A?

11.02. In the figure below,
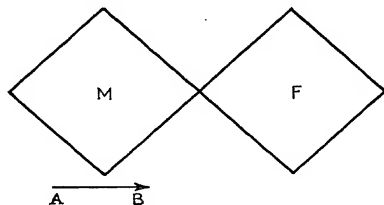
PROB. 11.02.

F represents 12 square, parallel, electrically connected metal plates, each having sides 3 cm. long and each having a thickness of 0.3 mm.; they are uniformly spaced with facing surfaces 1.1 mm. apart. M represents 11 plates, similar to those of F, mounted on a movable rack. The plates of M interleave the plates of F when M is moved parallel to AB. A pinion meshes with the rack of M, and when the pinion is turned through 360 deg. the plates M move parallel to AB a distance of $3\sqrt{2}$ cm. If a dial on the shaft of the pinion is marked in degrees, calculate the capacitance of the condenser formed by the two sets of plates for $0 \leq \theta \leq 360°$ and draw a calibration curve.

11.03. The reading of a charged electrostatic voltmeter decreases 15 per cent when an initially uncharged condenser is connected in parallel with the voltmeter. Calculate the capacitance of the condenser.

11.04. Two parallel No. 0000 A.W.G. wires 120 miles long are suspended 11 ft. apart and 25 ft. above the ground. Calculate the total charge on each wire if one wire is maintained at $+33,000$ volts and the other at $-33,000$ volts, with respect to ground. Calculate the capacitance per unit length and the forces on the wires.

11.05. If the plates of a parallel plate condenser are separated by a solid piece of material of dielectric constant ϵ/ϵ_0 and if the area of the plates is A square meters and their distance of separation is d meters, calculate the work required to pull the dielectric from the region between the plates, assuming the condenser to have been charged originally to a difference of potential V and then disconnected from the source of charge.

11.06. Calculate the force exerted on an uncharged sphere of radius R meters by a charge Q placed d meters from the center of the sphere. Repeat for a grounded sphere.

11.07. Calculate the force exerted on the charge Q of Problem 10.04.

11.08. Calculate the force per unit length exerted on the cylinder of Problem 10.05.

11.09. Refer to Problem 10.06. If a drop of water 1,400 m. above the surface of the earth has a net charge equal to that of 1 electron and if it remains at rest, calculate its diameter.

11.10. What is the capacitance per unit length of the concentric cylinders determined by the solution of Problem 10.11.

11.11. Calculate the force exerted on the charges Q of Problems 10.18 and 10.19.

11.12. Calculate the force on the charge Q_1 and the force on the charge Q_2 of Problem 10.21.

11.13. Calculate the 60-cycle susceptance per mile of each wire of a three-phase line of No. 0000 A.W.G. wire arranged at the apexes of an equilateral triangle each side of which is 6 ft. Check the result against data in an electrical engineers' handbook. Assume that $1/h < < < 1/d$, where h is the height of a wire above ground in feet and $d = 6$ ft.

11.14. If the plates of a "parallel"-plate condenser are not quite parallel, *i.e.*, if the maximum separation is $(s + a)$ and the minimum separation is $(s - a)$ and $a < < s$, show that the approximate capacitance is

$$C = \frac{\epsilon_0 S}{s}$$

11.15. If the inner sphere (radius R_1) of a concentric-spherical air condenser is coated with a thickness t of a material having a dielectric constant ϵ/ϵ_0, calculate the increase in capacitance.

11.16. If the inner sphere of a concentric-spherical condenser is slightly displaced, is the capacitance increased or decreased?

11.17. Prove that the capacitance of a spherical condenser with insulator of dielectric constant ϵ/ϵ_0 is equal to the capacitance between the same

spheres when half the space between them is filled with air and half with a liquid of dielectric constant $(2\epsilon/\epsilon_0 - 1)$.

11.18. Calculate the capacitance per unit area of a parallel-plate condenser in which the dielectric constant of the insulator is ϵ_1/ϵ_0 at one plate, ϵ_2/ϵ_0 at the other, and varies linearly between them. What is the volume polarization within the insulating material? What is the surface polarization at each plate?

11.19. The plates of a parallel-plate condenser are separated by a distance of 1 mm. Calculate the force per unit area on one of the plates if: (*a*) the plates are separated by air and the difference of potential between them is 100 volts; (*b*) the plates are charged to 100 volts, the battery is disconnected, and the space between the plates is then filled with oil having a dielectric constant of 3; (*c*) the condenser is again connected to a 100-volt battery.

11.20. Two condensers have capacitances C_1 and C_2. C_1 is charged by means of a battery, then disconnected from the battery and short-circuited. C_1 is then recharged and connected in parallel with C_2, producing a second spark. Finally C_1 and C_2 are disconnected from each other and short-circuited in the order C_1, C_2. Show that the energies of the four sparks are in the ratio

$$(C_1 + C_2)^2 : C_2(C_1 + C_2) : C_1^2 : C_1C_2$$

11.21. Two condensers are charged separately to V_1, V_2, and then connected in parallel. Calculate the loss of electrostatic energy and account for it.

11.22. A line charge of Q coulombs per meter is placed in a region of dielectric constant ϵ_1/ϵ_0, parallel to and at a distance a from the plane boundary between the region of dielectric constant ϵ_1/ϵ_0 and a region of dielectric constant ϵ_2/ϵ_0. Calculate the potentials in the two media and the force per unit length on the line charge.

11.23. A small sphere of radius r of material having a susceptibility χ is placed at a large distance d from a conducting sphere of radius R whose potential is V. Show that the dielectric sphere is attracted to the conducting sphere by a force F

if the susceptibility of the medium is negligibly small and if $r < < < d$.

11.24. Prove that the change in $\sqrt{C}$ is proportional to the change in angle of rotation; C being the capacitance of a rotary variable condenser of which the rims of the movable plates can be represented by $\theta/r^2 = $ constant.

11.25. What is the equation of the rims of the movable plates of a variable condenser for which the fractional change in capacitance is independent of the angle of rotation.

11.26. What are the expressions for the force on a charge Q placed at various points on a line through the center of an insulated and uncharged conducting sphere?

11.27. Calculate the capacitance per unit length and the force per unit length for two infinite conducting cylinders of radius R whose centers are d

m. apart and h m. above the earth if (a) the cylinders carry charges of $+Q$ and $-Q$ coulombs per m.; and (b) the cylinders are connected together and their combined charges are Q coulombs per m.

11.28. Prove that in the case of an electrostatically independent system of three conductors the total energy stored in the electrostatic field is

$$W \qquad\qquad\qquad + \tfrac{1}{2}C_{22}V_{2i}^2$$

where the V's are the potential drops from conductors No. 1 and No. 2 to the conductor of reference and the C's are the capacitances as defined in Eq. (11.11).

CHAPTER XII

ELECTROMAGNETIC FLUX
AND INDUCED ELECTROMOTIVE FORCES

12.01. Introduction.—The natural phenomena loosely described by the term *magnetic effects* are difficult to classify and to interpret, although many such effects are commonly observed. Thus the student will recall the phenomena described briefly below:

1. Pieces of magnetite (lodestone), a naturally occurring compound of iron, exert an attractive force on pieces of iron, cobalt, nickel, and certain alloys. Furthermore, a piece of steel, after being stroked by a piece of magnetite, possesses the power of attracting to *it* small pieces of these substances. The regions in which these effects are observed are said to be the sites of *magnetic fields* and the devices that appear to produce the fields are loosely called *magnets* (Gilbert[1]).

2. Electrically charged bodies in motion produce magnetic fields (Rowland[2]).

3. An electric current in a conductor produces a magnetic field. Therefore, current-carrying conductors exert mechanical forces upon each other (Oersted,[3] Ampere[4]).

4. If the magnetic field in the neighborhood of an electric conductor changes, an emf is in general induced in the conductor (Faraday[5]).

A quantitative theory of these phenomena can be developed from each of several different starting points. For example, all magnetic effects can be assumed to arise from the motion of electric charges (2 above). Since charges apparently do not move throughout a macroscopic volume of a steel magnet in any manner likely to produce the magnetic effects that are observed, the effects are tentatively attributed to subatomic entities. Experiments indicate that the external electrons and the atomic nucleus produce a magnetic effect like that which might be produced by small circular currents. The theory is then developed to predict what effects might be expected from atoms of

different elements and from aggregations of such atoms. The results are subjected to experimental test. This procedure[6] is to be recommended as the most fundamental now available, but it requires a comprehensive knowledge of the experimental and theoretical (quantum-mechanical) aspects of atomic physics. These aspects of atomic physics are not required for the analysis of most engineering problems, so that this logical and fundamental procedure is rejected in this book for engineering students in favor of a simpler, though less comprehensive, theory of magnetism.

A second possible starting point in the development of a theory of magnetism for engineers is the analysis of *magnetostatic fields* (1 above) in terms of *magnetic-field intensity* defined as the force per unit pole exerted on a very small north-seeking pole. This procedure is similar in some respects to the development of the theory of the electrostatic field presented in Chap. X, and it is commonly used in textbooks.[7] However, this procedure is here rejected both because an isolated magnetic pole is not available for experiment and because the concept of an isolated pole is deduced from experiments that are not important in engineering practice.

A third possible starting point is the force between current-carrying conductors.[8] This effect can be accurately measured, and it is important in electrical engineering practice.

A fourth possible starting point is the emf usually induced in a conductor that moves with respect to a magnetic field. This effect can be accurately measured, and it is important in electrical engineering practice. This starting point is used in this chapter and in Chaps. XIII and XIV. It is chosen in preference to the third chiefly because the quantities in terms of which electromagnetic fields are described can be more simply invoked from a discussion of induced emfs than from a discussion of mechanical forces between current-carrying conductors.

It is also possible to start by postulating the *vector potential* **A,** an integral similar in form to the integral used in Chap. X to define the electrostatic scalar potential. From this vector **A,** mathematical relations can be logically deduced; these relations are found to be in accord with measurements of magnetic phenomena. This procedure has several advantages. It is general; it is concise; and it accounts for phenomena associated

with current-carrying conductors, magnets, and electromagnetic fields.   It has one disadvantage if it be used as an *introduction* to the theory of magnetism: it is difficult for the student of an introductory course to correlate the experimental facts that he learns in the laboratory and the concepts associated with the vector potential.

In view of these facts, magnetic effects are discussed in this and subsequent chapters by stating experimental facts and then obtaining, *by a process of induction*, certain general mathematical relations.   In Appendix E, the magnetic vector potential is defined, and the mathematical theory of certain aspects of electromagnetism is obtained, by a *process of deduction.*

It is recommended that the student work through Chaps. XII to XIV and then study Appendix E.   He will probably find, after learning a little of the involved experimental basis of electromagnetic phenomena, that he can best coordinate his knowledge for future use by means of the deductions presented in Appendix E.

12.02. Faraday's Law; Magnetic Flux.—Throughout the remainder of this chapter certain useful quantities related to
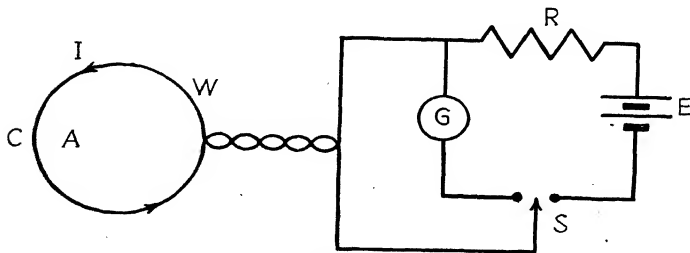
Fig. 12.01.

observed phenomena are defined and correlated.   The first quantity here defined is a scalar, the *magnetic* flux through a surface.*   Figure 12.01 shows a loop of wire W in the plane of the paper that can be connected by means of a twisted pair of insulated wires and a switch S to either a ballistic galvanometer G or a battery B.   When the switch S is closed to the battery, a current I flows in a counterclockwise direction in the loop W.   If a small compass needle is suspended in the interior area A, its

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* Called in the title of this chapter *electro*magnetic flux, to emphasize its electrical origin.

north pole points out from paper; if it is placed in the exterior
area C, its north pole points into the paper.   The orientation
of such a needle in the sense from its south pole to its north pole
is taken as an indication of the direction of a magnetic field at a
point.   The student will recall the use of such compass needles,
and of iron filings, for finding qualitatively the forms of magnetic
fields in the regions surrounding magnets and current-carrying
wires and coils of wire.

If the switch S is suddenly thrown (at $t = 0$) to the contact
connected to the ballistic galvanometer G, the current i does not
change instantaneously to zero but decreases to a small fraction
of its original magnitude I in a very small, but finite, interval of
time t.   The galvanometer G, whose deflection is assumed to be
proportional to the charge Q that passes through it in the time t,
indicates

$$(12.01) \qquad\qquad Q = \quad i\,dt \qquad\qquad \text{coulombs}$$

Subsequently no magnetic effects are observed in the region
surrounding the wire loop.*   The current i can be assumed to be
produced by an *induced emf e*.   Then, if the resistance and
inductance of the loop, leads, and galvanometer are R, L, it
follows from Eq. (12.01) that

$$(12.02) \qquad\qquad \int_0^t e\,dt = \int_0^t Ri\,dt + L\int_{i_{-0}}^{i_{t-t}} di$$

$$(12.03) \qquad\qquad Q = \frac{1}{R}\int_0^t e\,dt$$

if t is sufficiently large so that $i_{t=0} = i_{t=t} = 0$.   Originally
$(t = 0)$ the direction of the magnetic field was *out from* the paper
according to the convention described above; after the current
has decreased to a negligibly small value, there is no field into
the paper.

Magnetic flux ϕ through a surface is defined as the time integral
of the emf induced in a conductor forming the boundary of the
surface, throughout the interval during which the field is being
established.   This definition is

$$(12.04) \qquad\qquad = \pm \qquad\qquad = 0 \text{ at } t = 0)$$

* *If* the experimenter's apparatus is shielded from magnetic fields produced
by some other agent, such as the earth, and from all *varying electric fields*.

The choice of sign is determined by convention, particularly the conventions of vector analysis. For reasons discussed subsequently, the minus sign is chosen:

$$(12.05) \qquad \phi = - \int_0^t e\, dt \qquad (\phi = 0 \text{ at } t = 0)$$

The unit of measurement of magnetic flux is the *volt-second;* this unit has been named the *weber.*

Note that, since

$$(12.06) \qquad e = -L\frac{di}{dt} \qquad\qquad [\text{Eq. (4.01)}]$$

Eqs. (12.05) and (12.06) combine to produce

$$(12.07)[*] \qquad L = \frac{d\phi}{di} \qquad\qquad \text{henrys}$$

Since ϕ can conceivably be a function of variables other than i it is perhaps more specific to define inductance as

$$(12.08) \qquad\qquad \partial\phi \qquad\qquad \text{henrys}$$

Means based on Eq. (12.08) for calculating the inductance of transmission-line conductors and inductors of simple form are discussed in Chaps. XIII and XIV. Note that, if all the turns of a coil are not linked by the same flux, the self-inductance of the coil is the derivative with respect to the current of the total *flux linkages* $(N_1\phi_1 + N_2\phi_2 \cdots)$; N_1 turns being linked by the flux ϕ_1, N_2 turns by the flux ϕ_2, etc.

12.03. Electromotional-field Intensity.—In Chap. X, the electrostatic-field intensity was defined as the force per unit positive charge exerted on a small positive test charge placed in an electrostatic field in a region in which *there is no changing magnetic field.* The electrostatic-field intensity, represented below by the symbol $\mathbf{E}'$, has no circulation, *i.e.,*

$$(12.09) \qquad\qquad \mathbf{E}' \cdot \mathbf{ds} = 0$$

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[*] Since ϕ generally increases when i increases in any simple inductor, L is positive. If the plus sign had been chosen in Eq. (12.04), inductance would have been negative. Perhaps the convenience of having L positive is sufficient justification for the choice of the minus sign in Eq. (12.04). Further justification is discussed in Sec. 12.04.

In Sec. 12.02 the emf e induced in a loop of wire through which the magnetic flux is changing with time can conveniently be expressed as the closed-line integral, *i.e.*, the circulation, of the *electromotional-field intensity* $\mathbf{E}''$

$$\text{(12.10)} \qquad \oint \mathbf{E}'' \cdot \mathbf{ds} = e \qquad \text{volts}$$

This vector $\mathbf{E}''$ corresponds to the *electromotional force* discussed in Sec. 1.06.

It follows that the *electric-field intensity* $\mathbf{E}$ is the total force per unit charge at a point and that it is the vector sum of the electrostatic field intensity $\mathbf{E}'$ and the electromotional field intensity $\mathbf{E}''$. Since the circulation of $\mathbf{E}'$ is zero, it is common practice to write

$$\text{(12.11)} \qquad \mathbf{E} \cdot \mathbf{ds} = e \qquad \text{volts}$$

i.e., to use the general symbol $\mathbf{E}$ for the total electric-field intensity.

There is no need to consider only those values of the vector $\mathbf{E}$ that are related to points in a conducting loop. The most general theories of the electromagnetic field are expressed in terms of the vector $\mathbf{E}$ in free space, in insulators, and in conductors.

This general concept of the electric-field intensity is discussed further in the next section.

12.04. Magnetic-flux Density.—Magnetic flux through a surface is a scalar quantity. Yet the *direction* of a magnetic field has been specified. Apparently there is at least one *vector* associated with the field. One such vector point function is the *magnetic-flux density* $\mathbf{B}$, which is defined as the limit of the ratio of the flux through a small area perpendicular to the direction of the magnetic field, to the area, as the area approaches zero. The positive sense of the flux density $\mathbf{B}$ is the direction of the magnetic field at the point in question. The unit of flux density is the *weber per square meter*.

The flux ϕ through any closed path s is the surface integral of $\mathbf{B}$ over any *open* surface S_0 bounded by the path s:

$$\text{(12.12)} \qquad \phi = \int_{S_0} \mathbf{B} \cdot \dot{\mathbf{dS}}_0 \qquad \text{webers}$$

It follows, therefore, that the integral of *the outward flux* from a *closed* surface is zero; therefore lines representing $\mathbf{B}$ *are closed*

lines.   Application of Gauss's theorem (see Appendix B) results in

$$(12.13)* \qquad \int_{S_c} \mathbf{B} \cdot d\mathbf{S}_c = \int_v (\nabla \cdot \mathbf{B})\, dv = 0$$

in which S_c stands for a closed surface, such as a sphere.   For a small volume, Eq. (12.13) reduces to the differential equation:

$$(12.14)* \qquad\qquad\qquad \nabla \cdot \mathbf{B} = 0$$

This is one of the four relations commonly called Maxwell's equations.

Note that the electric–flux density **D** can be represented by lines that have their origins on *free* positive electric charges and their other terminal points on *free* negative electric charges. The lines representing the magnetic–flux density **B**, on the other hand, do not have beginnings and ends; they are closed loops. These two facts are represented mathematically by

$$(10.41) \qquad\qquad\qquad \nabla \cdot \mathbf{D} = \rho$$
$$(12.14) \qquad\qquad\qquad \nabla \cdot \mathbf{B} = 0$$

two of Maxwell's equations, which are discussed in their relation to electromagnetic radiation in Chap. XVI.   Note again that these equations are directly applicable only in regions where **D** and **B** are continuous functions.*

Another of Maxwell's equations can be derived here from the analysis given above and one more vector concept.   Given a vector **E**, there is related to it a vector $\nabla \times \mathbf{E}$ called "del cross **E**" or *the curl of* **E**.   This vector, in terms of coordinates x, y, z, takes the rather complicated form

$$(12.15)$$

A detailed discussion of the curl of **E** is given in Appendix B.   A vector **E** and its curl $\nabla \times \mathbf{E}$ are related in a manner useful in electromagnetic theory by Stokes's theorem, which is derived in Appendix B and illustrated in Fig. 12.02.

Given: (1) the values of **E** everywhere along a closed path s; and (2) the values of $\nabla \times \mathbf{E}$ everywhere on any open surface S_0 bounded by s.   Stokes's

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* Note that those expressions involving $\nabla \cdot \mathbf{B}$ are meaningless or require special analysis in regions where **B** is discontinuous (see Sec. 14.03 and Appendix E).

† The remainder of this section can be omitted without disturbing the continuity of this chapter and Chaps. XIII, XIV, and XV.   The remainder of this section can then be used as an introduction to Chap. XVI.

theorem states that the line integral of **E**, taken clockwise (when viewed from a point such as A) about the path s, is equal to the integral of $(\nabla \times \mathbf{E})$ over the open surface S_0:

$$(12.16) \qquad\qquad \oint \mathbf{E} \cdot \mathbf{ds} = \int_{S_0} (\nabla \times \mathbf{E}) \cdot \mathbf{dS_0}$$

The surface integral of Eq. (12.16) involved only *space* coordinates. Therefore, from Eqs. (12.05) and (12.11)

$$(12.17) \quad e = -\frac{\partial \psi}{\partial t} = -\,;$$

and from Eq. (12.16)

$$(12.18) \qquad\qquad \int_{S_0} (\nabla \times \mathbf{E}) \cdot \mathbf{dS_0} = - \int_{S_0} \frac{\partial \mathbf{B}}{\partial t} \quad \mathbf{dS_0}$$

which reduces, for a small area $\mathbf{dS_0}$, to the differential equation

$$12.19) \qquad\qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

This is a third Maxwell equation; the first two derived in this book being Eqs. (10.41) and (12.14). The fourth equation is derived in Sec. 12.06. In
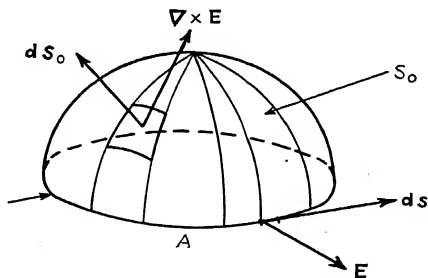
FIG. 12.02.

that section also, the four equations are listed in differential form, in integral form, and in the form of statements of the bases of their derivations.

12.05. Permeance, Reluctance, and Magnetomotive Force.—

It is the purpose of this section to discuss in terms of a hypothetical experimental procedure the magnetic field produced by a current in a conductor. Figure 12.03 shows a concentrated coil of wire C of N turns and a tubular volume M specified in a manner described below. A small circular test coil, connected by a twisted pair of wires to a ballistic galvanometer, is placed at a, in the plane of the coil. If the current in the coil C is

initially zero and rises quickly to I_1 amperes, the maximum deflection of the meter connected to the test coil is a measure of the total flux established through the test coil.   In the absence of ferromagnetic materials such as iron, cobalt, nickel, and certain alloys, *the flux ϕ' through such a test coil is found to be proportional to the current I_1 and the number of turns N of the coil C*

$$(12.20) \qquad\qquad \phi' = \mathcal{O}'NI_1 \qquad\qquad\qquad \text{webers}$$

where $\mathcal{O}'$ is the factor of proportionality having the dimensions weber per ampere turn; the factor $\mathcal{O}'$ is analyzed further below.
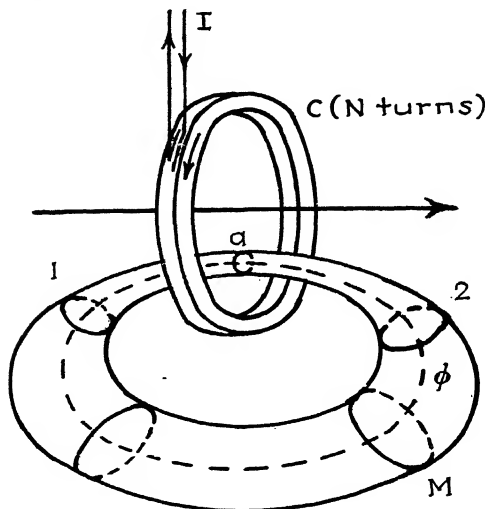
Fig. 12.03.

It is at least theoretically possible to move the test coil a short distance from a in the direction of the field at its center (*i.e.*, in the direction $a \rightarrow 2$, Fig. 12.03) and then to adjust the size and perhaps the shape of the test coil in such a manner that the flux through it is ϕ'.   If this procedure is repeated many times, the test coil sweeps out the surface of the tubular volume M.   This volume presumably encloses all of the flux ϕ' through any cross section of it, *i.e.*, no flux traverses its bounding surface.   Thus Eq. (12.20) applies to any cross section of M, and $\mathcal{O}'$ is apparently a factor depending upon the shape and size of the volume through

which ϕ' passes. The volume M could have been specified by recording the direction (from south pole to north pole) of a small compass needle moved about in the region of the coil C carrying a current I_1. The implication here is that the field may be thought of as a group of closed lines whose direction is everywhere the same as that indicated by a compass needle and whose density is everywhere proportional to the flux density. Then the quantity $\mathcal{P}'$ is related to that particular volume M which contains the closed lines representing ϕ'.

By using a test coil at a (Fig. 12.03) having larger or smaller cross section and by placing it at other positions than a in the plane of the coil C, volumes similar to M but of different lengths and cross sections can be specified by either of the two methods described above. The data of experiments similar to these suggest that:

1. The total flux ϕ produced by the current I in the coil C is proportional to the ampere turns NI

$$(12.21) \qquad\qquad \phi = \qquad\qquad\qquad\qquad \text{webers}$$

2. The quantity $\mathcal{P}$, called the *permeance* of the entire field* to the flux ϕ, can be constructed by summing the permeances $\mathcal{P}'$
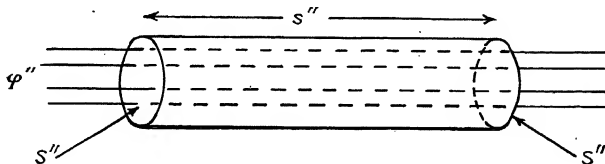
Fig. 12.04.

of all the regions similar to M. Furthermore, the experimental results are in accord with the assumption that each small volume (such as that shown in Fig. 12.04) of the space in which the field exists has a permeance $\mathcal{P}''$, which can be calculated as follows. First construct the small volume in the manner indicated in Fig. 12.04. Note that: (1) the flux ϕ'' through the volume is perpendicular to the surfaces S''; and (2) no flux enters or leaves the volume except through the end surfaces. Then $\mathcal{P}''$, the per-

* Which in this case theoretically occupies all space.

meance of the volume $S''s''$ is assumed to be

(12.22)   webers per ampere turn

in the absence of ferromagnetic materials. The quantity $\mu_0 = 1.257 \times 10^{-6}$ weber per ampere-meter (or henrys per meter) is called the *magnetic permeability* of free space. For any other medium a factor μ, the permeability of that medium, is to be used. The relative permeability μ/μ_0, a numeric, is approximately one for all substances except the ferromagnetic materials, which have permeabilities often much greater than one, and the diamagnetic materials, such as bismuth, which have permeabilities slightly less than one (see Chap. XIV).

The reciprocal of permeability is called the *reluctivity* and the reciprocal of permeance is called *reluctance* $\mathcal{R}$.

A magnetic field of any kind can presumably be considered as a large number of cells of the form shown in Fig. 12.04, and the permeances $\mathcal{P}_1''$, $\mathcal{P}_2''$, . . . , can presumably be combined to obtain either the permeance $\mathcal{P}'$ of a tubular volume such as M, Fig. 12.03, or to obtain the total permeance $\mathcal{P}$ for all of the region in which the field exists. These points are discussed further in the next section.

Note the following parallelism in the formulas derived above for a *magnetic circuit* and those discussed in Part 1 for an electric circuit:

For a cell such as that shown in Fig. 12.04 above

Reluctance $\mathcal{R}''$ and Permeance $\mathcal{P}''$ Resistance R_c and Conductance G_c

$$\mathcal{R}'' = \frac{s''}{}$$ $$R = \frac{\rho s''}{S''} = \frac{s''}{\gamma S''}$$

$$\mathcal{P}'' = \frac{}{s''}$$

For a circuit

$$NI = \mathcal{R}\phi$$ $$E = RI$$
$$\phi = \mathcal{P}NI$$ $$I = GE$$

Corresponding quantities

Reluctance of a cell, flux parallel to lateral walls	Resistance of a cell, current stream lines parallel to lateral walls
Permeance of a cell, flux parallel to lateral walls	Conductance of a cell, current stream lines parallel to lateral walls
Permeability and permeance	Conductivity and conductance
Reluctivity and reluctance	Resistivity and resistance
Flux	Current
Ampere turns	Electromotive force

The last items indicates the reason for calling the ampere turns the *magnetomotive force*. The essential lack of similarity between the two systems is the fact that a constant magnetic flux through a surface represents a *fixed state*, while a constant current through a surface represents the *motion* of charge at constant velocity through the surface.

12.06. Magnetic-field Intensity.—According to the assumptions described in Sec. 12.05, the reluctance $\mathfrak{R}''$ of a cell like that shown in Fig. 12.04 is

$$(12.23) \qquad \mathfrak{R}'' = \frac{s''}{\mu S''} \qquad \text{ampere turns per weber}$$

The flux ϕ'' through this cell is

$$(12.24)^* \qquad \phi'' = BS'' \qquad \text{webers}$$

Therefore

$$(12.25) \qquad \mathfrak{R}''\phi'' = \frac{Bs''}{\mu} \qquad \text{ampere turns}$$

The flux ϕ'' is enclosed in some volume such as M, Fig. 12.03. If this is divided into many sections, the sum of the products $\mathfrak{R}''\phi''$ is, according to Eq. (12.20), equal to the ampere turns *linked* by the flux

$$(12.26) \qquad \mathfrak{R}_1''\phi'' + \mathfrak{R}_2''\phi'' + \cdot \cdot \cdot = NI_1$$

The process on the left of Eq. (12.26) corresponds to an integration along the lines of flux. Thus putting $s'' = ds$ and $NI_1 = I_t$ the total equivalent current producing the flux, it follows from Eq. (12.25) that

$$(12.27) \qquad \oint \frac{B}{\mu}\,ds = I_t \qquad \text{ampere turns or amperes}$$

This relation is the basis of the definition of *magnetic-field intensity*† H:

$$(12.28) \qquad H = \frac{B}{\mu} \qquad \text{ampere turns per meter}$$

The field intensity is a vector; only its magnitude is involved in the discussion above because the path over which it is integrated

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* In general Eq. (12.24) should be written $\phi'' = \mathbf{B} \cdot \mathbf{S}''$; in this case the directions of $\mathbf{B}$ and $\mathbf{S}''$ are identical so that $\mathbf{B} \cdot \mathbf{S}'' = BS''$.

† Also called *magnetic-field strength and magnetizing force*.

is everywhere parallel to the field intensity.   Thus, in general,

(12.29) $\mathbf{H} = \dfrac{\mathbf{B}}{\mu}$ ampere turns per meter or amperes per meter

(12.30) $\oint \mathbf{H} \cdot \mathbf{ds} = I_t$ ampere turns or amperes

Note that, when the line of sight through the closed path s is in the direction of current flow, the integration is clockwise about the path s.

If the closed path s links no current, Eq. (12.30) reduces to

(12.31) $$\oint \mathbf{H} \cdot \mathbf{ds} = 0$$

Following the methods used in Chap. X, for relating the *vector* $\mathbf{E}$ and the *scalar* electrostatic potential V, a *magnetic scalar potential* U can be defined.   In the special case represented by Eq. (12.31)

(12.32) $$\mathbf{ds} = - \oint (\mathbf{\nabla} U) \cdot \mathbf{ds} = 0$$

and the *difference of magnetic scalar potential* between two points in a path that does not link any current is

(12.33) $\displaystyle\int_1^2 \mathbf{H} \cdot \mathbf{ds} = - \int_1^2 dU = U_1 - U_2 = U_{12}$ ampere turns

This is the *drop* in magnetic scalar potential from point 1 to point 2.   A surface over which the scalar magnetic potential is constant is a magnetic equipotential surface.   A volume bounded by sections of two equipotential surfaces, 1, 2, and a surface between them through which flux neither enters nor leaves, has a reluctance $\Re$.   The difference of magnetic potential between the surfaces is the *reluctance drop*

(12.34)   ampere turns

Although the concept of H as the negative gradient of the scalar magnetic potential U is useful, care must be used in applying the concept because U is multivalued along those parts of a path that "link" a current.   Thus the line integral of $\mathbf{H}$ around a closed loop that links a current I is not zero; the result is I.   This point is illustrated in the problem discussed below.   Note carefully the circumstances for which Eq. (12.32) is valid.

As a simple introduction to the use of the concept of magnetic scalar potential in a problem involving a path that *does* link current, consider Fig. 12.05. A cylindrical coil of wire of N turns has a current I amperes flowing in it. The magnetomotive force is therefore NI ampere turns. Consider now a tubular volume S whose axis is the dotted line s, Fig. 12.05. This volume contains flux ϕ webers. The crosshatched planes through S (3, 4) are magnetic equipotential surfaces. Accord-
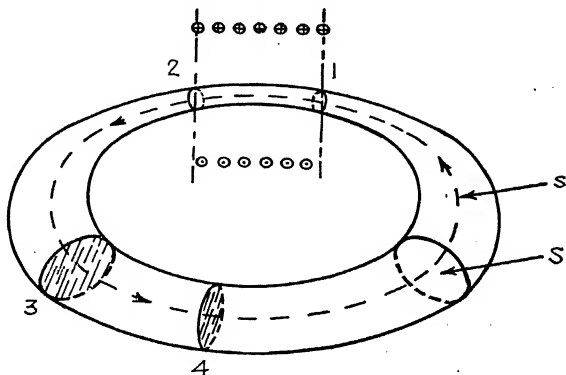
FIG. 12.05.

ing to Eq. (12.34) the *drop* in magnetic potential from 3 to 4 is

$$(12.35) \qquad\qquad U_{34} = \mathcal{R}_{34}\phi \qquad\qquad \text{ampere turns}$$

because the path $(3 \rightarrow 4)$ *does not* link any current. The magnetic potential *drop* from $1 \rightarrow 2$ comprises two parts: (1) the *drop* $\mathcal{R}_{12}\phi$ and (2) the *drop* $-NI$, *i.e.*, the magnetomotive force $+NI$, which represents a *rise* in magnetic potential. Therefore

$$(12.36) \qquad\qquad U_{12} = \mathcal{R}_{12}\phi - NI \qquad\qquad \text{ampere turns}$$

For the whole *magnetic circuit*, the total *drop* in magnetic potential along the path 2341 is equal to the *drop* in potential along the path 21

$$(12.37) \qquad U_{21} = \mathcal{R}_{2341}\phi = NI - \mathcal{R}_{12}\phi \qquad \text{ampere turns}$$

This equation has the same form as the expression for the difference of potential $V_2 - V_1$ between the terminals 2, 1 of a d-c

generator that has an internal resistance R_{12}, emf E, and load R_{2341}

$$(12.38) \qquad V_2 - V_1 = R_{2341}I = E - \qquad \text{volts}$$

Since the lines that represent the flux density **B** are closed lines, the net flux coming up to any surface is equal to the net flux leaving this surface; this is analogous to Kirchhoff's first law for electric circuits. Equation (12.37), or a generalized form of it, is analogous to Kirchhoff's second law.

The analogies between electric circuits and magnetic circuits discussed above and at the end of Sec. 12.05 are interesting, but the practical basis of calculations of magnetic circuits in electrical engineering is

$$(12.30) \qquad \oint \mathbf{H} \cdot \mathbf{ds} = I_t \qquad \text{ampere turns}$$

rather than the relations involving the concept of magnetic scalar potential.*

Returning now to a consideration of Eq. (12.30), note that by applying Stokes's theorem (see Appendix B):

$$(12.39) \qquad \oint \mathbf{H} \cdot \mathbf{ds} = \int_{S_0} (\nabla \times \mathbf{H}) \cdot \mathbf{dS_0}$$

The current I_t, Eq. (12.30), can be expressed as the surface integral of the current density J_t (a vector) over the surface S_0 bounded by the path of integration of **H**. Therefore

$$(12.40) \qquad \int_{S_0} (\nabla \times \mathbf{H}) \cdot \mathbf{dS_0} = \int_{S_0} \mathbf{J}_t \cdot \mathbf{dS_0}$$

The corresponding partial differential equation is

$$(12.41) \qquad \nabla \times \mathbf{H} = \mathbf{J}_t \qquad \text{amperes per square meter}$$

Here J_t is the *total* current density. If only the *conduction* current J is considered, this result is in accord with experiment only when there are in the region *no varying electric fields* (see Sec. 16.02). Note that $J = \rho \mathbf{u}$, in which **u** is the velocity of the charge $\rho\, dv$ in a volume dv. It is shown in Chap. XVI that, at a point where there is a moving charge $\rho\mathbf{u}$ and a changing electric field, $J_t = \rho\mathbf{u} + \partial\mathbf{D}/\partial t$. The relation in Eq. (12.41), modified to include the effects of the presence of varying electric fields, is the fourth Maxwell equation, derived in Chaps. X to XII. The table below shows the integral form, the differential form, and basis of derivation of each of Maxwell's equations.

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* The remainder of this section can be omitted without breaking continuity or it can be used as an introduction to Chap. XVI.

MAXWELL'S EQUATIONS FOR REGIONS IN WHICH **D, B, E,** AND **H** ARE
CONTINUOUS FUNCTIONS

Integral form	Differential form	Basis
$\int_{S_e} \mathbf{D} \cdot d\mathbf{S}_e = \int_v \rho \, dv$	$\nabla \cdot \mathbf{D} = \rho$	Inverse square law
$\int_{S_e} \mathbf{B} \cdot d\mathbf{S}_e = 0$	$\nabla \cdot \mathbf{B} = 0$	Faraday's law and definition of flux
$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \int_{S_0} \mathbf{B} \cdot d\mathbf{S}_o$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	Faraday's law
$\oint \mathbf{H} \cdot d\mathbf{s} = \int_{S_0} \mathbf{J}_t \cdot d\mathbf{S}_0$	$\nabla \times \mathbf{H} = \mathbf{J}_t = \rho \mathbf{u} + \frac{\partial \mathbf{D}}{\partial t}$	Inference from experiments of Oersted and Ampère

12.07. The Field Produced by a Straight Wire.—Although the
methods for calculating the magnetic-field intensity produced
by current-carrying conductors are the subject of Chap. XIII,
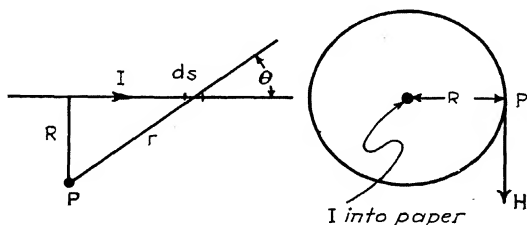
FIG. 12.06.

one simple problem is analyzed here in order to obtain one more
mathematical tool for these calculations.

Figure 12.06, *left*, shows a long straight wire, with constant
current I flowing in it.   Figure 12.06, *right*, shows the direction
of the field produced by the current-carrying wire.   By sym-
metry, H is constant in magnitude around a circle of radius R.
Therefore, from Eq. (12.30)

$$(12.42) \qquad \oint \mathbf{H} \cdot d\mathbf{s} = 2\pi R H = I$$

or

$$(12.43) \qquad = \frac{I}{2\pi R} \qquad \text{amperes per meter}$$

At the point P the vector $\mathbf{H}$ is perpendicular to the paper and it is directed downward, into the paper. Thus $\mathbf{H}$ is perpendicular to the wire and to lines joining P to any point on the wire. The question arises: Is there an expression for $d\mathbf{H}$, the field contributed by a short piece of wire of length $d\mathbf{s}$, which can be used to derive $\mathbf{H}$ by integration? The expression must satisfy for this particular problem the relation

$$(12.44) \qquad H_P = \int_{-\infty}^{\infty} F(s, I, R)\, ds = \frac{I}{2\pi R}$$

in which the function F is to be found. The problem is symmetrical, so that Eq. (12.44) can be written

$$H_P = 2 \int_0^{\infty} F(s, I, R)\, ds = \frac{I}{2\pi R}$$

or

$$\int_0^{\infty} F(s, I, R)\, ds = \frac{I}{4\pi R}$$

Note from Fig. 12.06 that to change from the variable s to the variable θ

$$s = \frac{R \cos \theta}{\sin \theta} \qquad ds = -\frac{R}{\sin^2 \theta}\, d\theta$$

$$s \to \infty \qquad \text{corresponds to} \qquad \theta \to 0$$

$$s = 0 \qquad \text{corresponds to} \qquad \theta = \frac{\pi}{2}$$

Therefore

$$(12.45) \quad \int_0^{\infty} F(s, I, R)\, ds = \int_0^{\frac{\pi}{2}} F(\theta, I, R) \frac{R}{\sin^2 \theta}\, d\theta = \frac{I}{4\pi R}$$

One arrangement that will satisfy Eq. (12.45) is

$$(12.46) \qquad F_1(\theta, I, R) = \frac{I \sin^3 \theta}{4\pi R^2}$$

so that $\displaystyle \int_0^{\frac{\pi}{2}} F_1(\theta, I, R) \frac{R}{\sin^2 \theta}\, d\theta = \frac{I}{4\pi R} \int_0^{\frac{\pi}{2}} \sin \theta\, d\theta = \frac{I}{4\pi R}$

Substituting for ds, Eq. (12.44) becomes, r being the distance from the wire to point P,

$$(12.47) \qquad H_P = \int_{-\infty}^{\infty} \frac{I \sin \theta\, ds}{4\pi r^2} \qquad \text{amperes per meter}$$

A possible assumption, valid in this case at least, is that the contribution to the magnetic field dH_P at a point P, produced by a current I in an elementary line conductor ds is that given below, Eqs. (12.48) and (12.49), and illustrated in Fig. 12.07, *left*

$$(12.48) \qquad dH_P = \frac{I \sin \theta \, ds}{4\pi r^2} \qquad \text{amperes per meter}$$

Note that if $\mathbf{r}_1$ is a unit vector in the direction of r, the general
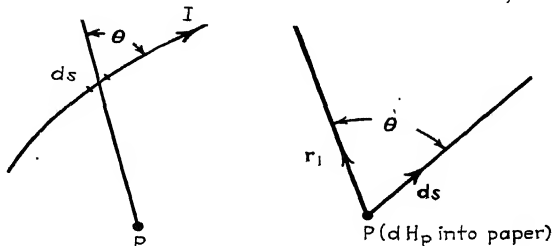
Fig. 12.07.

vector relation that describes the magnitude *and* direction of dH_P at P is (see Fig. 12.07, *right*):

$$(12.49)^* \qquad \mathbf{dH}_P = \frac{I}{4\pi} \frac{\mathbf{r}_1 \times \mathbf{ds}}{r^2}$$

The discussion in this section is in no way a derivation or proof of Eq. (12.48). The relation in Eq. (12.49) can be derived from the concept of the vector potential (see Appendix E). This useful relation is one of several[8] firmly founded on the experiments performed by Ampère during the third decade of the nineteenth century. These experiments are not discussed further, but the relation in Eq. (12.49) is used in Chap. XIII to calculate the fields produced by various configurations of current-carrying conductors.

References

1. *On the Magnet;* William Gilbert (1544–1603); Chiswick Press, London, 1900.

* This is often written

$$\frac{I}{4\pi} \frac{\mathbf{ds} \times \mathbf{r}_1'}{r^2}$$

where $\mathbf{r}_1'$ is a unit vector in the direction from $\mathbf{ds}$ to P.

2. *The Mathematical Theory of Electricity and Magnetism;* J. H. Jeans; pp. 514–515, Cambridge University Press, London, 1923.
3. *A Treatise on Electricity and Magnetism;* J. C. Maxwell; 3d ed., Vol. II, p. 138, Clarendon Press, Oxford, 1904.
4. See 3, p. 141.
5. *Faraday's Diary;* Michael Faraday; Vol. 1, pp. 367 *et seq.*, George Bell & Sons, Ltd., London, 1936.
6. *Modern Magnetism;* L. F. Bates, Cambridge University Press, London, 1939.
7. *Introduction to Theoretical Physics;* Leigh Page; 2d ed., pp. 409–425, D. Van Nostrand Company, Inc., New York, 1935.
8. *Principles of Electricity and Electromagnetism;* G. P. Harnwell; pp. 275*ff.*, McGraw-Hill Book Company, Inc., New York, 1938.

Problems

12.01. A 100-turn, plane, concentrated coil enclosing an area of 20 sq. in. is placed in a region in which there is a uniform magnetic field, and connected by means of a twisted pair to a cathode-ray oscillograph. The plane of the coil is originally perpendicular to the direction of the field. When the coil is removed from the field, the oscillograph shows that the instantaneous emf in the coil varies with time as follows:·

Time, seconds	Emf, volts
0.000	0
0.002	15
0.004	30
0.006	47
0.008	65
0.010	73
0.012	73
0.014	60
0.016	27
0.018	10
0.020	0

Calculate the flux density of the magnetic field.

12.02. A conductor in the form of a rectangle of length a and width h rotates with an angular velocity ω radians per second about an axis parallel to a and passing through the mid-points of h. The axis is coincident with a line $x = 15$ in a rectangular coordinate system; there is a magnetic field perpendicular to the x axis and to the loop axis whose magnitude is $B = A\varepsilon^{2x}$. Calculate the emf induced in the conductor as a function of time.

12.03. A concentrated coil of 20 turns is linked by a flux ϕ webers. The ends of the wire of the coil are connected by means of a twisted pair to a Duddell oscillograph; the resistance of the entire circuit is 25 ohms. When the coil is removed from the field, the oscillogram is found to be representable by

$$0.04t^2 - 0.4t + 0.16i^2 = 0 \qquad (i \text{ ma., } t \text{ sec.})$$

Calculate the flux ϕ.

12.04. The flux density along the air gap of an a-c machine is

$$B = B_1 \sin x + B_3 \sin 3x$$
$$B_1 = 3B_3 = 0.2 \qquad \text{weber per m.}^2$$

Design a coil that, when moving at a velocity of 100 m. per sec. through this field, will have induced in it an emf of maximum value 50 volts with no third harmonic.

12.05. A thin disk of copper 12 in. in diameter rotates at an angular velocity of 2,000 rpm; its plane is perpendicular to a uniform magnetic field and the total flux through the disk is 0.015 weber. What is the difference of electric potential between the edge of the disk and the center of the disk?

12.06. The axial length (or thickness) of a concentrated circular coil of N turns is negligible in comparison with its diameter. If the current flowing in the coil is I amperes, (a) what is the difference of magnetic potential between the two faces of the coil; (b) what is the reluctance drop from one face of the coil to the other along any path which does not thread the coil; (c) what is the reluctance drop from one face to the other through the coil?

12.07. A concentrated rectangular coil of N turns of insulated wire has a mean length of a m. and a mean width of b m. It is rotated about an axis that intersects the sides of length b at their mid-points at a constant angular velocity of ω radians per second. The axis is perpendicular to a constant magnetic field of density B webers per square meter. The ends of the coil of wire are connected. The coil has inductance L and resistance R. Calculate the voltage induced in the coil, the current that flows in the coil, the average power dissipated by the coil, and the position of the coil with respect to the field B when the current is maximum.

12.08. A power line PP and a telephone line TT are suspended on poles like that shown. If the current in the power line is 300 amp., 60 cycles, calculate the maximum voltage induced in the telephone line.
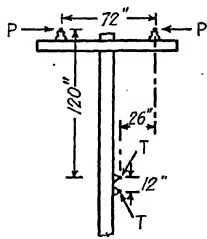
PROB. 12.08.

CHAPTER XIII

THE CALCULATION OF MAGNETIC-FLUX DENSITY; INDUCTANCE

13.01. Magnetic-flux Density Produced by Electric Current.—
The elementary theory of the production of magnetic fields by
currents has been developed in Chap. XII and Appendix E.
It is the purpose of this chapter to use the tools there developed
to calculate the fields **H** and the flux densities **B** produced by a
few commonly used configurations of conductors and to calculate
their self-inductances. The problems here considered are those
in which the fields traverse only paramagnetic substances
($\mu = \mu_0$). The complications arising from the use of ferro-
magnetic materials are discussed in Chap. XIV.

The flux density **dB** produced at a point P by an elementary
length **ds** along which a current I flows is, from the generalization
of Ampère's experiments,

(13.01)   from Eq. (12.49)

Knowing that **dB** is perpendicular to both the element **ds** and
the line r, it is often possible to determine the *direction* of the
flux density at the point in question
by considering the configuration and
particularly the symmetry of the con-
ductors made up of many elements **ds**.
It is then necessary only to calculate
the *magnitude* of the flux density and
so to avoid the complexity of a vector
integration.
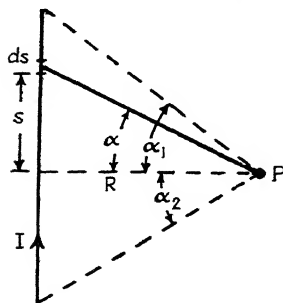
**13.02. Flux Density Produced by
Current in a Straight Wire.**—The flux
density produced by a current I in
a straight wire, shown in Fig. 13.01,
can be calculated directly from Eq. (13.01).* The magnitude
of dB, produced at P by ds, is

FIG. 13.01.

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* By reversing the process discussed in Sec. 12.07.

$$(13.02) \qquad dB = \frac{\mu_0 I}{4\pi} \frac{\sin \theta \, ds}{r^2}$$

and it is directed into the paper at P. Since $\sin \theta = \cos \alpha$ and $ds = R \, d\alpha / \cos^2 \alpha$, the flux density B is

$$(13.03) \qquad B = \frac{\mu_0 I}{4\pi} \int_{-\alpha_2}^{\alpha_1} \frac{\cos \alpha}{R} \, d\alpha = \frac{\mu_0 I}{4\pi R} (\sin \alpha_1 + \sin \alpha_2)$$

webers per square meter

Thus for a very long wire $\alpha_1 = \alpha_2 \approx \pi/2$, and Eq. (13.03) reduces to

$$(13.04) \qquad B = \frac{\mu_0 I}{2\pi R} \qquad \text{webers per square meter}$$

13.03. Flux Density within a Long Cylindrical Conductor.— It is assumed in Sec. 12.07 that the current I is confined to a line. Actually, of course, a wire carrying a current has stream lines of current distributed more or less uniformly over its cross section, depending upon the rate of change of current in the wire and upon the presence of other current-carrying conductors in the vicinity of the conductor being considered. Note that the differential equations developed in Chap. XII are linear for paramagnetic materials or for any materials for which μ is a constant. Therefore, the flux density **dB** produced by one elementary "piece" of current can be added to that produced by another—just as the currents in a linear circuit can be added and the contributions of the electrostatic potential at a point can be added. The flux densities must, however, be added *vectorially*.

Consider now the hypothetical case of a very long cylindrical conductor of radius r_0 carrying a constant current I with no return conductor. If this conductor is far removed from other conductors and magnets, and if the current does not vary with time, the current is uniformly distributed over the cross section of the conductor. The current density is therefore a constant J

$$(13.05) \qquad J = \frac{I}{} \qquad \text{amperes per square meter}$$

The flux density B outside the conductor is

$$(13.06) \qquad B = \qquad \text{webers per square meter for } R \geq r_0,$$

as shown in Fig. 13.02.   The flux density B_r within the conductor, at distance r from its center is produced by the fraction of the total current that flows in a cylinder of area πr^2

$$(13.07) \quad B_r = \frac{\mu_0}{2\pi r} \frac{\pi r^2}{\pi r_0^2} I = \frac{\mu_0 I}{2\pi r_0^2} r \qquad \text{webers per square meter for}$$

Thus the flux density at the center of the conductor is zero.
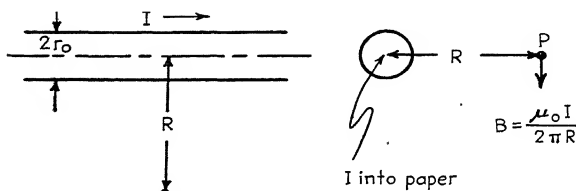
FIG. 13.02.

13.04. Inductance per Unit Length of Long Cylindrical Conductor.—The inductance per unit length of a long cylindrical conductor *with uniform current density* can be calculated directly from Eqs. (13.06) and (13.07) and the definition of inductance

$$(13.08) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{henrys}$$

although the result is of little practical significance for reasons discussed below.   The inductance L comprises two parts, L_1—the contribution by flux that links all the current I, and L_2—the contribution by the flux linking a fraction of the current, within the conductor.   Calling x the distance from the center of the conductor to the surface of a cylinder of radius x and thickness dx, the first part is

$$(13.09) \quad L_1 = \frac{\partial}{\partial I} \left[\int_{r_0}^{A} (B \times 1) \, dx \right] = \frac{\partial}{\partial I} \left[\int_{r_0}^{A} \frac{\mu_0 I}{2\pi x} \, dx \right]$$

$$= \frac{\mu_0}{2\pi} \log \frac{A}{r_0} \qquad\qquad\qquad \text{henrys per meter}$$

in which A is a distance large enough so that further increase in x causes an inappreciable change in flux linkages.   The mathematician would call A *infinity*.   Thus the relation in Eq. (13.09) has no practical use unless the arrangement of conductors in a

practical problem is such that another term $- log A$ appears in the analysis to remove the $+ log A$ term of Eq. (13.09).

The part contributed by flux within the wire can be conveniently calculated by using a relation between two quantities: (1) the energy related to an inductor carrying a current, from the point of view of circuit theory; (2) the energy of the field produced by the inductor, from the point of view of field theory. Assume that the flux density at any point, produced by a current in an inductor, is directly proportional to the current; *i.e.*, assume $\mu = \mu_0$ throughout the field. If the self-inductance of the inductor is L henrys, and the current at time t is i amperes, the following relations are satisfied.

Self-induced emf $\qquad\qquad\qquad e = -L\dfrac{di}{dt}$

Current in inductor	i	amperes
Power input to inductor	$-ei$	watts
Energy input in time dt	$-ei\,dt = Li\,di$	joules

Magnetic flux at time t $\qquad\qquad\qquad\qquad\qquad$ webers

Increase in flux in time dt $\qquad\qquad di \qquad\qquad$ webers

Inductance in terms of flux and current $\quad L = \dfrac{a\varphi}{di} \qquad$ henrys

Energy input in time dt $\qquad\qquad\qquad d\phi \qquad$ joules

If the current is caused to increase from $i = 0$ at $t = 0$ to $i = I$ at $t = t$, the flux will increase from $\phi = 0$ at $t = 0$ to $\phi = \phi$ at $t = t$; the result of integration is

13.10) $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ joules

Therefore, if means can be found for evaluating the quantity $\Re\phi^2$, L can be calculated from Eq. (13.10).

The part L_2 of the inductance per unit length of a straight cylindrical conductor contributed by flux within the wire is calculated below by using the relation of Eq. (13.10). A piece of the conductor, 1 meter in length, is represented by the drawing, Fig. 13.03. The flux density at radius r is, from Eq. (13.07)

(13.11) $\qquad\qquad\qquad B = \qquad r \quad$ webers per square meter

The flux $\Delta\phi$ through the strip of width Δr and length 1 meter is

$$(13.12) \qquad \Delta\phi = B\,\Delta r = \frac{\mu_0 I}{2\pi r}\,r\,\Delta r \qquad\qquad \text{webers}$$

The path of this flux is a cylinder of radius r and radial thickness Δr; the length of this path is $(2\pi r)$, and the cross section is $(1 \times \Delta r)$. Therefore the reluctance $\Delta\mathfrak{R}$ of the path of this flux
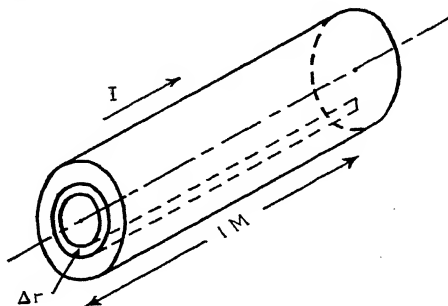
Fig. 13.03.

$\Delta\phi$ is, from Eq. (12.23)

$$(13.13) \qquad \Delta\mathfrak{R} = \frac{2\pi r}{\mu_0\,\Delta r}$$

It follows that the energy $d(\tfrac{1}{2}\mathfrak{R}\phi^2)$ associated with a cylinder of radius r, length 1 meter, and thickness dr is

$$(13.14) \qquad d(\tfrac{1}{2}\mathfrak{R}\phi^2) = \lim_{\Delta r \to 0}\,[\tfrac{1}{2}(\Delta\mathfrak{R})(\Delta\phi)^2]$$

$$= \frac{\mu_0 I^2}{4\pi r_0^4}\,r^3\,dr \qquad\qquad \cdot\ \text{joules}$$

Integrating this result from $r = 0$ to $r = r_0$ and substituting in Eq. (13.10) produces the value of L_2.

$$(13.15) \qquad\qquad L_2 = \frac{\mu_0}{8\pi} \qquad\qquad \text{henrys per meter}$$

Therefore the total self-inductance per meter of a long cylindrical conductor carrying a current that is uniformly distributed over its cross section is

$$(13.16) \qquad\qquad L = \frac{\mu_0}{2\pi}\left(\frac{1}{4} + \log\frac{A}{r_0}\right) \qquad \text{henrys per meter}$$

In any practical circuit (which is always a *closed* circuit), to this self-inductance must always be added the mutual inductance due to the so-called *return current* in the rest of the circuit.

13.05. Concentric Cylindrical Conductors—Uniform Current Density.—Suppose that a constant current I flows along a cylindrical conductor of radius r_0 and back through a concentric hollow cylindrical conductor of inner radius r_1 and outer radius r_2, i.e., of thickness $(r_2 - r_1)$ as shown in Fig. 13.04. First note
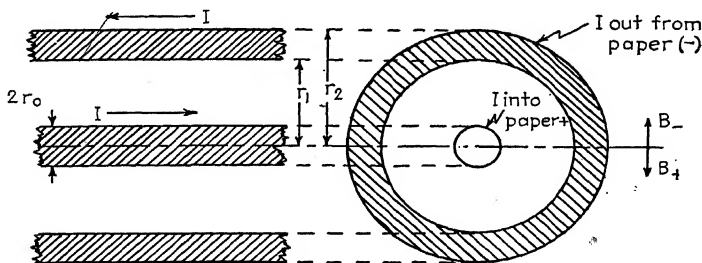
Fig. 13.04.

that there is no magnetic field outside the conducting system, because

$$\oint \mathbf{H} \cdot \mathbf{ds} = I - I = 0$$

so that $\mathbf{H} = 0$ and $\mathbf{B} = \mu_0\mathbf{H} = 0$ for $r \geq r_2$

This result can also be derived from the fact that $\mathbf{B}$ at a distance $x(x > r_2)$ consists of two equal and opposite components B_+

$$\cdot = \frac{\mu_0 I}{2\pi x}$$ produced by the inner conductor

and

$$-\frac{\mu_0(-I)}{2\pi x}$$ produced by the outer conductor.

Therefore the total effective inductance of 1 meter of the combination (central conductor and concentric return conductor) can be calculated in terms of the flux linkages produced by each of the conductors separately. The contribution L' of the inner conductor is from Eq. (13.16)

(13.17) $$L' = \frac{l}{2} \qquad \frac{\cdot\cdot}{\cdot\cdot} \Big|$$ henrys per meter

The contribution L'' of the outer conductor comprises two terms also:

$$(13.18) \qquad L_1'' = \frac{\partial}{\partial I}\left[\int_{r_2}^{A}(B'' \times 1)\,dx\right] = \frac{\mu_0}{2\pi}\log\frac{A}{r_2}$$

where A has the meaning discussed above, and

$$(13.19)\quad L_2'' = \frac{\mu_0}{8\pi}\left[\frac{r_2^2 - 3r_1^2}{r_2^2 - r_1^2} + \frac{4r_1^4}{(r_2^2 - r_1^2)^2}\log\frac{r_2}{r_1}\right]\ \text{henrys per meter}$$

The total effective inductance L is the sum $(L' - L_1'' - L_2'')$

$$(13.20)\quad L = \frac{\mu_0}{2\pi}\left[\log\frac{r_2}{r_0} + \frac{r_1^2}{2(r_2^2 - r_1^2)} - \frac{r_1^4}{(r_2^2 - r_1^2)^2}\log\frac{r_2}{r_1}\right]$$
$$\text{henrys per meter}$$

It often happens that $(L_2' - L_2'')$ is so much smaller than $(L_1' - L_1'')$ that a practically satisfactory formula is

$$(13.21)\qquad\qquad L \approx \frac{\mu_0}{2\pi}\log\frac{r_2}{r_0}\qquad\qquad\text{henrys per meter}$$

13.06. Skin Effect.—If the current in the inner conductor of a concentric cable changes with time, the current density is *not*

FIG. 13.05.

constant over the cross section of the conductor. The current density is greater on the outside surface of the conductor. This effect—called the *skin effect*—is often (but not always[1]) negligible from a practical point of view for a-c frequencies less than a few thousand cycles per second; it is extremely important at ultra-high frequencies ($> 50 \times 10^6$ cycles per second). It is the purpose of this section to show how the magnitude of the current density can be determined. The derivation of a differential equation for the current density as a function of the distance from the center of the conductor and of time is an interesting and straightforward example of the practical application of the concepts described in Chap. XII. The process of solving this equation is not very interesting and certainly not straightforward. Therefore the equation is derived below in detail, the process of solving it is treated summarily and superficially, and a special
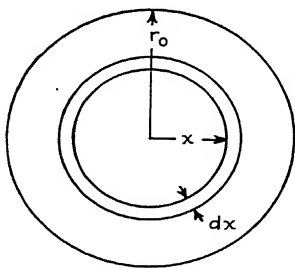
case of the solution is discussed in detail. Figure 13.05 shows the cross section of a cylindrical central conductor of a concentric cable. It is assumed that a sinusoidal alternating current of frequency f cycles per second flows parallel to the axis of the cylinder (perpendicular to the paper in Fig. 13.05). The problem is cylindrically symmetrical with respect to the axis of the conductor. Note the following symbols:

B flux density perpendicular to the radius at distance x from the axis.
ϕ flux *within* the conductor linking unit length of a cylinder of radius x.
ϕ' flux *outside of* the conductor linking unit length of conductor.
ρ resistivity of the material of the conductor.
μ_0 permeability of the material of the conductor.
v voltage drop along the conductor per unit length.
i_x instantaneous current in cylinder of radius x.
J current density at any point in the cylinder—J is everywhere parallel to the axis of the cylinder, all in mks units.

Since J is presumably a function of x, the flux density B, x meters from the axis, cannot be written in the form of Eq. (13.07) but must be left in the form

$$(13.22) \qquad B = \frac{\mu_0}{2\pi x} \int_0^x J 2\pi x \, dx \quad \text{webers per square meter}$$

Since ϕ is defined as the flux *within* the conductor that links a cylinder of radius x, this flux decreases $(-\Delta\phi)$ when x increases $(+\Delta x)$, and the amount of decrease, for unit length of the conductor, is $B \times 1 \times \Delta x$. Therefore , as $\Delta x \rightarrow 0$

$$(13.23) \qquad \frac{\partial \phi}{\partial x} = -B = -\frac{\mu_0}{2\pi x} \int_0^x J 2\pi x \, dx$$

The second partial derivative of ϕ with respect to x and to t is

$$(13.24) \qquad \frac{\partial^2 \phi}{\partial x \, \partial t} = -\frac{\mu_0}{x} \int_0^x x \frac{\partial J}{\partial t} \, dx$$

The voltage drop per unit length v consists of three parts:
1. The resistance drop, which is ρJ volts per meter.
2. The drop produced by the change in flux within the conductor, which is $\partial \phi / \partial t$.
3. The drop produced by the change in external flux, which is

Next note that $\partial \phi' / \partial t$ does not vary with x within the conductor

and that all sections of the conductor perpendicular to its axis are assumed to be equipotential surfaces. It follows then that

$$(13.25) \qquad \frac{\partial v}{\partial x} = 0 = \rho \frac{\partial J}{\partial x} + \frac{\partial^2 \phi}{\partial x \, \partial t}$$

Combining Eqs. (13.24) and (13.25)

$$(13.26) \qquad x \frac{\partial J}{\partial x} = \frac{\mu_0}{\rho} \int_0^x x \frac{\partial J}{\partial t} \, dx$$

Taking the partial derivative of Eq. (13.26) with respect to x

$$(13.27) \qquad \frac{\partial^2 J}{\partial x^2} + \frac{1}{x} \frac{\partial J}{\partial x} = \frac{\mu_0}{\rho} \frac{\partial J}{\partial t}$$

The solution of Eq. (13.27) is derived below as an example of methods; the results in Eqs. (13.48) to (13.54) can be examined without studying their derivation, if desired. The solution of this equation is begun by assuming that the current density $J(x, t)$ is equal to the product of a function $X(x)$ of x only and a function $T(t)$ of t only. Thus, substituting

$$(13.28) \qquad J = X \times T$$

in Eq. (13.27), there results

$$(13.29) \qquad T \frac{\partial^2 X}{\partial x^2} + \frac{T}{x} \frac{\partial X}{\partial x} = \frac{\mu_0 X}{\rho} \frac{\partial T}{\partial t}$$

or

$$(13.30) \qquad \frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{xX} \frac{\partial X}{\partial x} = \frac{\mu_0}{\rho T} \frac{\partial T}{\partial t}$$

Now the left-hand member involves functions of x and their derivatives while the right-hand member involves only t. If Eq. (13.30) is valid for all x within $0 \leq x \leq r_0$ and for all t, each member must be equal to the same constant. Furthermore, if the right-hand member is equal to a real number —positive, negative, or zero—the solution is absurd from the physical point of view. Therefore, the constant to which both members of Eq. (13.30) are equal is an imaginary number jk^2 where k^2 is real. Putting the right-hand member of Eq. (13.30) equal to this constant, and using the total derivative in place of the partial derivative,

$$(13.31) \qquad \frac{dT}{T} = \frac{jk^2\rho}{\mu_0} \, dt \qquad \text{or} \qquad T = A_1 \varepsilon^{j\frac{k^2\rho}{\mu_0}t}$$

in which A_1 is the constant of integration. If the current through the conductor is sinusoidal of frequency f cycles per second, the quantity $(k^2\rho)/(\mu_0)$ must be equal to $2\pi f$. Therefore

$$(13.32) \qquad k = \left(\frac{2\pi\mu_0 f}{\rho}\right)^{\frac{1}{2}} \qquad \left(\text{dimensions } \frac{1}{\text{meters}}\right)$$

There remains the problem of solving

$$(13.33) \qquad \frac{d^2X}{dx^2} \cdot \frac{dX}{dx} - jk^2xX = 0$$

This is a special form of a more general equation that has been investigated in some detail:[2]

$$(13.34) \qquad x_1^2 \frac{d^2y}{dx_1^2} + x_1 \frac{dy}{dx_1} + (x_1^2 - n^2)y = 0$$

This equation is known as Bessel's equation. The modified form, Eq. (13.33), is obtained from Eq. (13.34) by substituting $x_1 = j^{\frac{1}{2}}kx$ and $n = 0$. The solution of Eq. (13.33) is a complex quantity involving two infinite series. Since these series converge quickly for many values of the argument that are practically useful, their numerical values have been tabulated.[3,4] Therefore the function $ber(kx)$ (often called *Bessel real*) and $bei(kx)$ (often called *Bessel imaginary*) defined below may be considered simply as functions like sine and cosine whose values can be obtained from tables. The solution of Eq. (13.33) is

$$(13.35) \qquad X = A_2[ber(kx) + jbei(kx)]$$

in which

$$(13.36)* \quad ber(kx) \equiv 1 - \frac{1}{(2!)^2}\left(\frac{kx}{2}\right)^4 + \frac{1}{(4!)^2}\left(\frac{kx}{2}\right)^8 - \frac{1}{(6!)^2}\left(\frac{kx}{2}\right)^{12} + \cdots$$

and

$$(13.37)* \quad bei(kx) \equiv \left(\frac{kx}{2}\right)^2 - \frac{1}{(3!)^2}\left(\frac{kx}{2}\right)^6 + \frac{1}{(5!)^2}\left(\frac{kx}{2}\right)^{10} + \cdots$$

Combining Eqs. (13.31) and (13.35) according to Eq. (13.28) the current density $J\varepsilon^{j2\pi ft}$ is

$$(13.40) \quad J\varepsilon^{j2\pi ft} = A_1A_2[ber(kx) + jbei(kx)]\varepsilon^{j2\pi ft} \quad \text{amperes per square meter}$$

The complex number I representing the total current in the conductor is

$$(13.41) \qquad I = 2\pi \int_0^a Jx\,dx$$

$$I = 2\pi A_1A_2 \int_0^a x[ber(kx) + jbei(kx)]\,dx$$

The integrals are those of Eqs. (13.38) and (13.39) so that

$$(13.41a) \qquad I = \frac{2\pi A_1A_2r_0}{k}[bei'(kr_0) - jber'(kr_0)]$$

---

* If $ber'(kx) \equiv \dfrac{d}{dx}[ber(kx)]$ and similarly for $bei'(kx)$, prove that

$$(13.38) \qquad \int x\,ber(kx)\,dx = \frac{x}{k}bei'(kx)$$

$$(13.39) \qquad \int x\,bei(kx)\,dx = -\frac{x}{k}ber'(kx)$$

Next calculate the complex number V, representing the voltage drop per unit length along the surface of the conductor *excluding the drop produced by external flux:*

$$(13.42) \qquad \rho J_{r_0} =$$

The power input per unit length P is the real part of the product of V and the conjugate of I

$$(13.43) \quad P = \frac{}{\kappa} \qquad\qquad \text{watts per}$$

$$\text{meter}$$

From Eq. (13.41a) the magnitude of I^2 is

$$(13.44) \qquad I^2 = \frac{4\pi^2 A_1^2 A_2^2 r_0^2}{\lambda^2} [bei'^2(kr_0) + ber'^2(kr_0)]$$

Then R_f, the resistance per unit length at frequency f, is the ratio of Eq. (13.43) to Eq. (13.44). Calling the terms in the square brackets of Eq. (13.43) $P(kr_0)$ and the terms in the square brackets of Eq. (13.44) $I^2(kr_0)$,

$$(13.45) \qquad R_f = \frac{\rho k}{2\pi r_0} \frac{P(kr_0)}{I^2(kr_0)} \qquad \text{ohms per meter}$$

The resistance of the conductor to direct current is

$$(13.46) \qquad R_0 = \frac{\rho}{\pi r_0^2} \qquad \text{ohms per meter}$$

so that

$$(13.47) \qquad \frac{R_f}{R_0} = \frac{kr_0}{2} \frac{P(kr_0)}{I^2(kr_0)}$$

This result is useful only if tables of $P(kr_0)$ and $I^2(kr_0)$ are available.

An approximation for relatively high frequencies has been derived by Rayleigh[5]

$$(13.48) \qquad \overline{R_0} \qquad\qquad \frac{1}{4} +$$

Thus if $kr_0 \geq 71$, the first term above gives an answer which has an error of less than 2 per cent. If $kr_0 \geq 5$, the first two terms gives an answer accurate to less than 2 per cent. Putting the constants for copper in Eq. (13.32),

$$(13.49) \qquad kr_0 = 21.4 r_0 \sqrt{f}$$

Thus for a wire 1 centimeter in diameter and a frequency of 10,000 cycles per second, $kr_0 = 21.4$ and Eq. (13.48) becomes

$$(13.50) \qquad \frac{kr_0}{2\sqrt{2}} + \frac{1}{4} = 7.82.$$

If the diameter of the wire is 0.1 millimeter and the frequency is 100 megacycles, the value of $kr_0 = 21.4$ and Eq. (13.48) gives the same ratio as that shown by Eq. (13.50). Thus the skin effect in a small conductor is less

at a given frequency than it is in a larger conductor.   Note that in all cases
for which it is practicable, a tubular conductor is approximately as good a
conductor of high-frequency currents as a solid cylindrical conductor of the
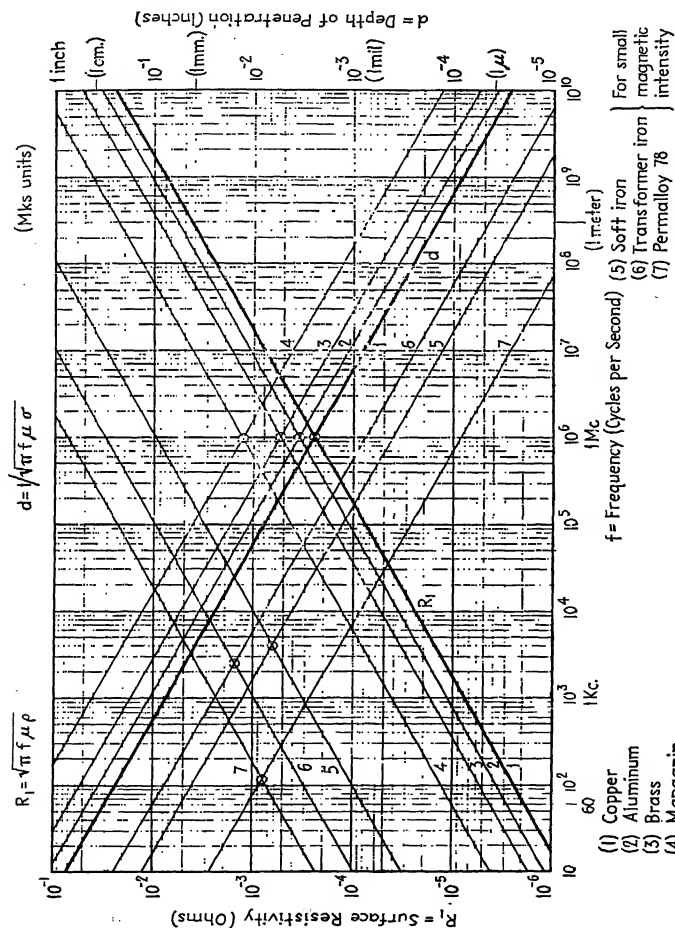
FIG. 13.06.—Skin effect, surface resistivity, and depth of penetration.   (Reproduced from *Formulas for the Skin Effect*; Harold A. Wheeler; *Proc. I.R.E.*; **30**, 9, 412–424, September, 1942; by permission of the author and the Institute of Radio Engineers.)

same diameter, and the hollow conductor is less costly and lighter in weight.
Two facts, not discussed above, can be derived from a more detailed analysis
of the solutions of Eq. (13.33): (1) the phase of the current density J with

respect to the voltage drop per unit length V varies with depth; (2) the phenomena of the skin effect cause the inductance per unit length to vary slightly with frequency.

If the skin effect is pronounced, *i.e.*, if nearly all the current flows in a cylinder whose outer radius is r_0 and whose inner radius is $(r_0 - \delta)$, where $\delta << r_0$, the calculation of the over-all effect is greatly simplified. Note the following results of such an analysis; the details are given in one of the references.[6] If the magnitude of the current density at the surface of a conductor is J_{r_0}, the *depth of penetration* or *skin depth* is defined as the depth δ at which the current density is $J_{r_0}/\varepsilon = 0.368J_{r_0}$. When skin effect is pronounced,

$$(13.51) \qquad \qquad \delta = \qquad \qquad \text{meters}$$

which is for copper

$$(13.52) \qquad \qquad \delta = \frac{0.066}{\sqrt{f}} \qquad \qquad \text{meters}$$

The *surface resistivity* R_1 for pronounced skin effect is defined as the (*dc*) resistance of a cube, each side of which is δ meters long:

$$(13.53) \qquad \qquad R_1 \qquad \qquad \text{ohms}$$

which, from Eq. (13.51) is

$$(13.54) \qquad \qquad \qquad \qquad \text{ohms}$$

Values of R_1 and δ for several conductors are shown in the chart, Fig. 13.06.

13.07. Total Inductance per Meter of Long Parallel Cylindrical Conductors.—The exact calculation of the inductance per meter of a transmission line is complicated. The following difficulties arise:

1. If the two wires of the line carry equal currents in opposite directions, a mechanical force tends to force them apart, so that they may not be parallel throughout their lengths.*

---

* The magnetic field in either wire also produces a slight nonuniformity of current density over the cross section of the wire, regardless of the frequency of the current that produces the field. Two phenomena—the *Hall effect*[7] and the *pinch effect*[8]—contribute to this result, but this nonuniformity is negligibly small in most important practical cases.

2. Skin effect causes a nonuniform distribution of current density over the cross sections of the conductors.

3. If the conducting materials are ferromagnetic, the relative permeability of the material may vary from point to point in the conductor.

4. The supporting towers and other near-by conductors may introduce factors difficult to calculate.

Fortunately the approximate results that are calculated by assuming that the conductors are straight, cylindrical, paramagnetic, and parallel, and that the current density in them is uniform over their cross sections, are accurate enough for practical purposes. The results of such calculations are particularly useful for the analysis of power transmission lines (25 cycles and 60 cycles). The approximate formulas from which tabulated
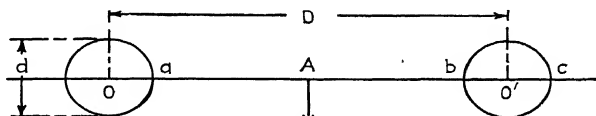
Fig. 13.07.

data[9] of the inductance and inductive reactance per unit length are calculated are derived in this section.*

A cross section of the line is shown in Fig. 13.07. The wires are d meters in diameter and their centers are separated by a distance D meters. The current I is assumed to be uniformly distributed over the cross sections and to flow in the directions into the paper in the left-hand wire and out of the paper in the right-hand wire. Thus the flux cutting the line OO' is in the direction of the arrow shown at A.

The inductance per unit length *per wire* is given by Eq. (13.16)

$$(13.55) \quad L = \frac{\mu_0}{2\pi}\left[\frac{1}{4} + \log\frac{2A'}{d}\right] \qquad \text{henrys per meter per wire}$$

where A' is the distance from O to b, O', or c; the choice of one of these remains to be discussed. Clearly all the flux from a to b links the current I in the left-hand wire. Flux produced solely by the left-hand wire, crossing the horizontal line OO' to the right of c links no current $(I - I)$. Flux produced solely by

*See also Appendix E.

the left-hand wire in the region from b to c links current $(I - kI)$, where k is zero at b and 1 at c. Therefore, as a tentative approximation, A' may be put equal to D, the value which is halfway between $D + d/2$, which is clearly too large, and $D - d/2$, which is clearly too small. The proof that $A' = D$ leads to an "exact" value of L is given in one of the references.[9]

The inductance of a pair of parallel cylindrical paramagnetic wires is, then,

$$(13.56) \quad L = \frac{\mu_0}{2\pi} \left[\frac{1}{4} + \log \frac{2D}{d} \right] \qquad \text{henrys per meter per wire}$$

$$(13.57) \quad L' = \frac{\mu_0}{\pi} \left| \frac{\cdot}{4} \right. \qquad \text{henrys per meter}$$

subject to all the qualifications stated in this section.

13.08. Flux Density in Terms of a Solid Angle.—The vector integration of Eq. (13.01) for any very complicated arrangement of conductors is often difficult. However the relation in Eq. (13.01) can be interpreted in terms of solid geometry in a manner
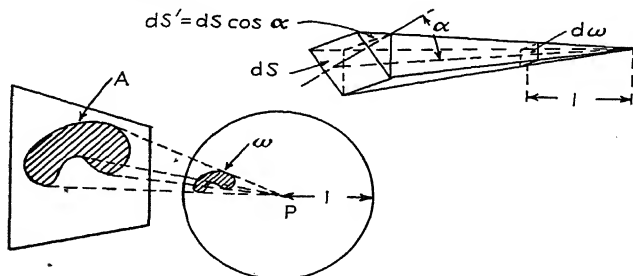
Fig. 13.08.

that simplifies greatly the calculation of the flux density produced by many commonly used configurations of conductors. It is the purpose of this section to introduce this interpretation and of the following sections to apply it.

The radial projection of an area of any size and shape on the surface of a sphere of unit radius is called the *solid angle* subtended at the center of the sphere by the area. The solid angle ω subtended by the area A is shown, with construction lines, by the example, Fig. 13.08, *left*. The projection of a small area dS upon the unit sphere produces the small solid angle $d\omega$ as shown

in Fig. 13.08, *right*. If the line from the point P makes an angle α with the perpendicular to dS, the projection dS' of dS upon a sphere of radius x is

(13.58) $$dS' = dS \cos \alpha$$

and the projection of this area dS' upon the surface of the sphere of unit radius is

(13.59) $$d\omega = \frac{dS'}{x^2}$$

Therefore the solid angle subtended at P by a surface S is the integral over the surface S of

(13.60) $$d\omega = \frac{\cos \alpha \, dS}{x^2}$$

Suppose that the element ds of a conductor carrying a current I is assumed to be displaced an infinitesimal distance $d\beta$ per-
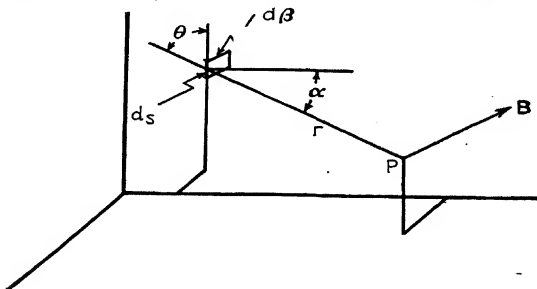
Fig. 13.09.

pendicular to the line r from P to ds, and to ds, *i.e.*, in the direction of the flux density B, Fig. 13.09. The area swept out by this assumed displacement is $ds \, d\beta$, and the solid angle subtended by it at P is

(13.61) $$d\omega = \frac{\cos \alpha (ds \, d\beta)}{-} = \frac{\sin \theta \, ds}{-} \, d\beta$$

Comparing this to Eq. (13.02) it appears that

(13.62)* $$dB = \pm \frac{\mu_0 I}{4\pi} \frac{d\omega}{d\beta}$$

* Note that if ω_0 is the solid angle subtended at P, Fig. 13.09, *by a surface bounded by a closed circuit*

(13.63) $$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = -\frac{I}{4\pi} \boldsymbol{\nabla}\omega_0$$

This relation may be used to define the *magnetic scalar potential* U, which is

If the displacement $d\beta$ had been assumed to be in some other direction than perpendicular to the element ds and the line r, the calculation by means of Eq. (13.62) would have given the component of **dB** at P in the direction of the displacement if the current were vertically *upward* in the element ds [plus sign of Eq. (13.62)], Fig. 13.09, and in the opposite direction if the current were downward [minus sign of Eq. (13.62)]. Thus the procedure is general, and it is particularly useful for getting a rough estimate of the magnitude of B at a point, without carrying
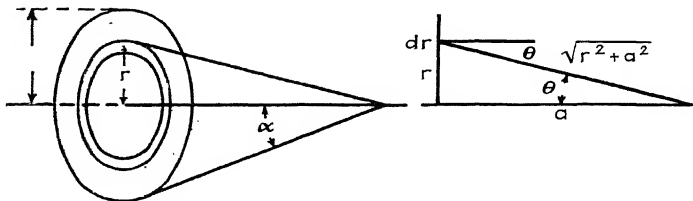
Fig. 13.10.

out the exact calculations. Examples are worked out in the following sections.

Note that the solid angle subtended at a point in a plane surface by the surface is 2π. This can be proved as follows. The solid angle subtended by a circular surface at a point along the line perpendicular to its plane at its center can be calculated as indicated in Fig. 13.10, and the following relations:

$$(13.65) \qquad d\omega \qquad \frac{2\pi r \cos\theta \; dr}{r^2 + a^2} \qquad \frac{2\pi r a}{} \; dr$$

$$(13.66) \quad \omega = 2\pi a \int_0^R \frac{r}{(r^2 + a^2)} \; dr = 2\pi a$$

$$= 2\pi(1 - \cos\alpha)$$

analogous to the electrostatic potential V

$$(13.64) \qquad\qquad U = \frac{I\omega_0}{4} \qquad\qquad \text{amperes}$$

This quantity can be used to derive the formulas developed in some of the remaining sections of this chapter (see also Sec. 12.06). Note particularly that, as was pointed out on p. 316, ω_0 is multivalued along those parts of a path that link a current. Therefore, although Eq. (13.63) may be used to calculate the field at a point distant from a circuit, care must be used to avoid errors where ω_0 is multivalued.

Thus if the point P lies in the plane of the circle, $\alpha = \pi/2$ and $\omega = 2\pi$

13.09. The Flux Density on the Axis of a Concentrated Circular Coil.—Assume that a current I amperes flows in a circular conductor as indicated in Fig. 13.11.   The flux density $\mathbf{B}$ produced at the point P is in the direction OP, from symmetry.   If the conductor is displaced a distance $d\beta$ parallel to OP as indi-
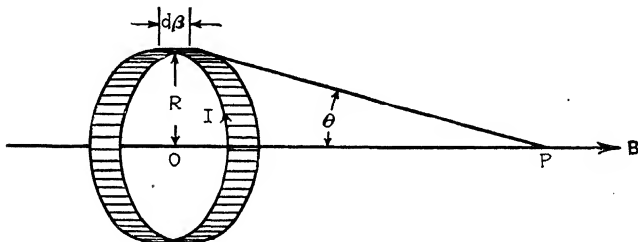
Fig. 13.11.

cated, it sweeps out the cylindrical area that is crosshatched in the figure.   The solid angle $d\omega$ subtended by this surface at P is

$$(13.67) \qquad d\omega = \qquad \sin\theta \; d\beta = \frac{2\pi \sin^3\theta \; d\beta}{R}$$

Therefore, from Eq. (13.62), the flux density B is

$$(13.68) \qquad B = \frac{\mu_0 I}{2R} \sin^3\theta \quad \text{webers per square meter}$$

Note that it is assumed that the current-carrying conductor is equivalent to a hypothetical line current, so that Eq. (13.68) is an approximation.   Similarly an N-turn concentrated coil produces a flux density

$$(13.69) \qquad B = \frac{}{2R} \sin^3\theta \quad \text{webers per square meter}$$

if $(r^2 + R^2)^{\frac{1}{2}}$ is large compared to the diameter of the cross section of the concentrated coil.

13.10. Solenoid.—A closely wound, single-layer coil of wire, forming a cylinder, is called a *solenoid*.   A cross section of a solenoid through its axis is shown in Fig. 13.12.   The marks $\oplus$ indicate that the current flows away from the reader in the wires

at the top of the figure; the marks ⊙ indicate that the current flows toward the reader in the lower wires.   If there are N turns of wire and the solenoid is displaced λ/N meters to the left, a cylindrical surface of R meters radius and of length λ is swept out by the wire.   Since from symmetry the flux density **B** at a point *on the axis* of the coil is parallel to the axis, its magnitude can be expressed as a function of the solid angle subtended by the cylinder at the point.   For example, at the point P

(13.70) $B =$

in which ω_1 and ω_2 are the solid angles subtended by the circular
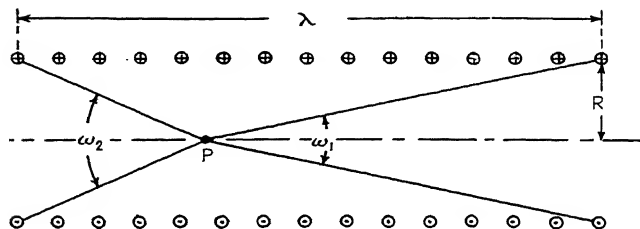
Fig. 13.12.

ends of the solenoid at P.   At the point halfway between the end surfaces of the solenoid

(13.71) $\omega_1 = \omega_2 = 2$

from Eq. (13.66).   Therefore at the center of the solenoid

(13.72) $B =$ webers per square meter

When $\lambda/2 \gtrsim 10R$, the term R^2 in Eq. (13.72) can be neglected to obtain an approximate result, which is in error by 0.5 per cent at most:

(13.73) B $\lambda \gtrsim 20R$

This is equivalent to the assumption that, in Eq. (13.70),

$$(\omega_1 + \omega_2) <<< 4\pi.$$

If P is in one end surface of the solenoid, $\omega_2 = 2\pi$ and

$$\lambda$$

For this case

(13.74) $\qquad B \qquad\qquad \overline{\lambda^2)^{\frac{1}{4}}}$ webers per square meter

and the approximation for a long solenoid is

(13.75) $\qquad B \qquad\qquad 2\lambda \qquad\qquad \lambda \geq 20R$ (60)

i.e., one-half as large as the flux density at the center of the solenoid.

The inductance of a circular coil or of a solenoid can be calculated by means that are not discussed in this book. These methods are based upon the principles discussed in this chapter. They involve the use of special mathematical tools, such as elliptic integrals, and of carefully chosen approximations; these are not suitable material for an elementary course.[8]
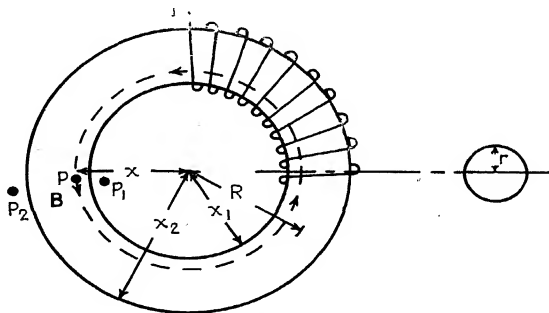
FIG. 13.13.

13.11. Toroid.—A toroidal winding with circular cross section is shown in Fig. 13.13. The wire is assumed to be wound closely over the entire surface of the doughnut. If the current I flows through N turns, the intensity of the magnetic field at P or at any other point on the circle of radius x can be calculated as follows

(13.76) $\qquad\qquad \mathbf{H} \cdot d\mathbf{l}$

so that

$$(13.77) \qquad \frac{NI}{2\pi x} \qquad \text{webers per square meter}$$

Thus B clearly varies over the cross section of the toroid.   Also note that the flux density at points such as P_1 or P_2 is (nearly)* zero, *i.e.*, the flux is confined to the core of the toroid.

If the cross-sectional diameter ($2r$, Fig. 13.13) of the core of the toroid is small compared to the overall diameter ($2x_2$, Fig. 13.13) of the toroid, the flux density is *approximately* constant for $x_1 \leq x \leq x_2$ so that the total flux is approximately

$$(13.78)\dagger \qquad \phi \approx \frac{r^2 \mu_0 NI}{2R} \qquad \text{webers}$$

and the inductance of the toroidal winding is

$$(13.79)\dagger \qquad L \approx \frac{r^2 \mu_0 N^2}{2R}. \qquad \text{henry}$$

The relation in Eq. (13.79) follows from the definition of inductance as the partial derivative of flux linkages ($N\phi$) with respect to the coil current (I).

In the case of an air-core toroid for which $r << R$, the energy W of the magnetic field is, from Eq. (13.79)

$$(13.80) \qquad W = \frac{1}{2} LI^2 = \frac{1}{2}\left(\frac{\mu_0 NI}{2\pi R}\right)\left(\frac{NI}{2\pi R}\right)(2\pi R)(\pi r^2) \qquad \text{joules}$$

The first quantity in parenthesis is the flux density B, the second is the magnetizing force H, and the product of the last two is the volume in which the magnetic field exists.   Therefore, the energy per unit volume W' of the magnetic field is

$$(13.81) \qquad W' = \frac{1}{2} BH = \frac{1}{2}\frac{B^2}{\mu_0} = \frac{1}{2}\mu_0 H^2 \quad \text{joules per cubic meter}$$

---

* Note that the coil is equivalent, from a point of view external to it, to a single turn carrying a current I.   Therefore, the flux density at a point such as P_1 or P_2 is not zero; it is small compared to the flux density *within* the toroidal winding.

† The approximations of Eqs. 13.78 and 13.79 must be used with care. For a discussion of the magnitudes of the errors involved in their use see *Principles of Electrical Engineering;* W. H. Timbie and Vannevar Bush; 3d ed., pp. 376–381, John Wiley & Son, Inc., New York, 1940.

Compare Eq. (13.81) with the expression for the energy per unit volume of an electrostatic field, Eq. (11.69). Both Eqs. (11.69) and (13.81) are used in the discussion in Chap. XVI of the propagation of energy in an electromagnetic field.

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11. *Formulas and Tables for the Calculation of Mutual and Self-inductance;* E. B. Rosa and F. W. Grover; *Bull. Bur. Standards;* **8,** 1–237; 1912 (*Bur. Standards Paper* 169).

Note: An interesting controversial discussion is given in *A New Factor in Induction; the "Loop" versus the "Cutting Lines of Force" Laws;* Carl Hering; *Elec. World,* Mar. 14, 1908, and in papers referred to by the author.

Problems

13.01. Calculate the flux density produced by a current I flowing in a conductor that forms a regular plane polygon of n sides inscribed in a circle of radius R, at the center of the circle. Show that, as n increases without limit, the flux density approaches that produced by a circular current.

13.02. A concentrated circular coil of 100 turns 15 cm. in diameter carries a current of 10 amp. At the center of the coil is a small concentrated coil of 10 turns 1 cm. in diameter; this coil is rotated at an angular velocity of 3,000 rpm. Calculate the emf induced in the smaller coil.

13.03. Calculate the mutual inductance between a 1,000-turn solenoid 50 cm. long and 2 cm. in diameter, and a 100-turn coil wound over the solenoid near its center.

13.04. Calculate the flux density produced by the current in a solenoid of radius R and length λ at a point on a line perpendicular to the axis of the

solenoid, distant d from the axis $(d > R)$. What is the approximation for B if $d >> \lambda$?

13.05. An air-core solenoid of 1,000 turns carrying a current of 1 amp. is 1 m. long and 0.1 m. in diameter.

   a. What size enameled copper wire is used for the coil?

   b. What is the terminal voltage of the solenoid?

   c. What is the total power dissipated by the coil?

   d. Calculate the flux density at the center of the solenoid. By what percentage does it differ from $B = \mu_0 NI/\lambda$?

   e. What are the internal and external reluctance drops along a tube near the axis of the solenoid?

   f. What is the difference of magnetic potential between the ends of the solenoid?

   g. What is the flux density at any point 20 cm. from the axis of the solenoid along a line perpendicular to the axis at the center of the solenoid?

13.06. A sphere of radius R has a uniform surface-charge density σ and it is rotated about a diameter at a uniform angular velocity ω radians per second. Calculate the flux density at the center of the sphere.

13.07. Prove that the flux density is uniform throughout the volume of the sphere of Problem 13.06.

13.08. All conductors of length a have resistances of r ohms except that marked x. If the voltage at AB is E volts, calculate the resistance of the
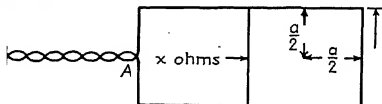
Prob. 13.08.

middle wire x ohms for the condition that the flux density at P shall be independent of E.

13.09. Calculate the self-inductance of a concentrated rectangular coil of N turns and sides a and b $(a > b)$.

13.10. Two currents I and I' flow in two parallel-plane conductors formed into two squares each having sides a. The line joining their centers is perpendicular to both squares and its length is c. Calculate the maximum flux density along this line if the currents flow in the same direction.

13.11. Design the concentrated circular coil using 100 ft. of No. 28 A.W.G. copper wire that will produce maximum flux density on the axis of the coil at a distance of 1 ft. from the coil when the coil is connected to a 6-volt storage battery.

13.12. Calculate the flux density at the center of a sphere produced by a uniform single-layer winding over the sphere for two cases: (*a*) uniform distribution along a diameter; (*b*) uniform distribution over the surface.

13.13. Calculate the mutual inductance between a long parallel-wire circuit (separation $2a$) and a circular conductor of radius b $(< a)$ placed in the plane of the parallel conductors midway between them.

13.14. A long straight wire carrying a current $i = 300 \sin 2\pi 60t$ intersects the plane of a toroid perpendicularly at its center. The circular cross section of the toroid has a diameter of 2 cm.; the mean major diameter is 8 cm.; and the uniform toroidal winding is 475 turns of No. 28 A.W.G. wire. Calculate the voltage induced in the toroidal winding.

13.15. Calculate the flux density produced by a current I in a conductor forming a square of sides a at a point b meters from the plane of the square on a line perpendicular to the plane of the square at one of the corners of the square.

13.16. A fine insulated wire is wound in a close spiral forming a disk of radius r. Show that the flux density at a point a meters from the disk on its axis, when a current I flows in the wire, is

$$B = \tfrac{1}{2}\mu_0 n I (\cosh^{-1}(\sec D) - \sin D) \qquad \text{webers per square meter}$$

n being the number of turns per meter measured along the radius and $D = \tan^{-1}(r/a)$.

13.17. Calculate the mutual inductance between two coaxial circular coils of radii a and b separated by a distance c. What is the approximate formula when $c \gg a$ and $c \gg b$?

13.18. Calculate the variation of inductance with frequency of the inner conductor of a concentric cable.

FERROMAGNETISM; POTENTIAL ENERGY AND FORCES ASSOCIATED WITH THE MAGNETIC FIELD

14.01. Introduction.—The last two chapters were concerned chiefly with the definitions of quantities such as magnetic-field intensity, flux density, and reluctance, and with their correlations by means of experimental data and mathematics. In order to simplify the presentation of this material, certain simplifying assumptions such as the constancy of permeability were made. Actually, many important electromagnetic engineering devices[13] depend for their effectiveness and economy upon the use of magnetic materials whose permeabilities are far from constant. Furthermore, the mechanical forces between current-carrying conductors, and between current-carrying conductors and magnets are often the most important practical features of such apparatus, and these phenomena have not yet been discussed. Therefore it is the purpose of this chapter to discuss briefly the underlying physical aspects, both experimental and theoretical, of the mechanical forces produced by magnetic fields and of the phenomena that occur when such magnetic materials as iron are placed in a magnetic field. Since the theory of ferromagnetism is both complicated and incomplete, the criterion for choosing the material of this chapter is neither completeness nor rigor; it is applicability to the solution of a few important engineering problems. It is important first to review the qualitative theory of magnetic effects in material media, mentioned briefly in the introduction to Chap. XII (Sec. 12.01).

Experiment shows that when a magnetic field is established in a material medium the ratio of the magnetic flux density B at any point in this medium to the magnetizing force H at this point, *viz.*, the ratio $\mu = B/H$ in general differs from the magnetic permeability μ_0 of free space. For certain substances the ratio μ/μ_0 is slightly (a few parts in a million) *less* than unity; such substances are called *diamagnetic* substances. For other sub-

stances the ratio μ/μ_0 is slightly (a few parts in a thousand) *greater* than unity; such substances are called *paramagnetic* substances.   There are a very few substances, such as iron, nickel, cobalt, and certain alloys, for which the ratio μ/μ_0 is many times unity, even as high as a million; such substances are called *ferromagnetic* substances.   From the engineering point of view one usually refers to both diamagnetic and paramagnetic substances as *nonmagnetic* and considers as a *magnetic substance* only those for which the relative permeability μ/μ_0 is, under normal conditions,* appreciably greater than unity.

It can be inferred from experiment that all magnetic effects are attributable to the motions of electric charges.   It was proposed by Ampère, nearly a century before the Bohr theory of the atom was initiated largely by the study of spectroscopic data, that minute constant circular currents in iron were the causes of the enhancement of magnetic effects that accompanies the introduction of pieces of iron into a magnetic field.   The difficulty of explaining how such hypothetical *amperian* currents could be maintained in iron was partly balanced by the implications that: (1) the known magnetic phenomena seemed, at least superficially, to be in accord with Ampère's theory; and (2) electromagnetic phenomena required for their correlation only electric charges, at rest or in motion; the concept of magnetic poles could be added for convenience, but it was unnecessary.   Unfortunately this simplified theory was held in abeyance until atomic physicists showed that it is useful to assume the presence in atoms of very small circular currents of just the kind that Ampère had suggested.   During this century, indirect experimental evidence of such subatomic currents has led to the general acceptance of theories of magnetism based upon theories of the structure of the atom.   Unfortunately the quantitative development of these theories requires a thorough knowledge of atomic physics, as noted in Sec. 12.01.

The classical theory of ferromagnetism postulated small circular currents, of which the *magnetic moments* were defined as outlined below.   The poles of permanent magnets were then interpreted as the effects of many such currents, pulled into line by other magnets or current-carrying conductors.   In the

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* Dependent upon the previous history of the substance, there may be certain values of B for which $\mu = B/H$ is zero (see Fig. 14.03).

development of this theory the magnetic pole was taken as the starting point. Thus current-carrying conductors were represented by *magnetic shells* that were imaginary parallel surfaces; one surface of the shell was assumed to have a uniform distribution of magnetic north poles, the other a uniform distribution of south poles; the shell was bounded by the conductor.

Although this theory is now seldom used, some elementary aspects of it are given below and in Sec. 14.04 for two reasons: (1) these classical ideas are aids to the understanding of the phenomena of ferromagnetism; (2) the excerpts from the classical theory that are presented in this chapter show clearly the interesting analogous development of the classical theory of ferromagnetism and the theory of dielectric polarization discussed in Sec. 10.12.

A small circular current can be represented schematically as indicated in Fig. 14.01, *left*. The magnetic scalar potential U, at a point P far removed
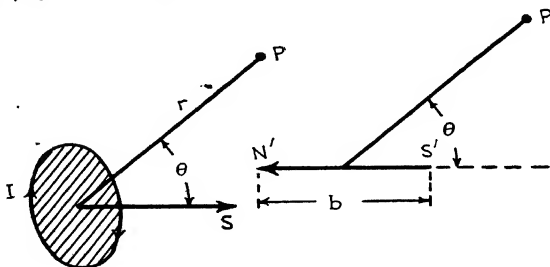
FIG. 14.01.

from the current, produced by this constant current is (see footnote, page 339)

$$(14.01) \qquad U \quad \frac{I}{4} \frac{\mathbf{S} \cdot \mathbf{r_1}}{r^2} \qquad \text{ampere (turns)}$$

A quantity called the *magnetic moment* $\mathbf{p}_m$ of such a current is defined as

$$(14.02)* \qquad \mathbf{p}_m = \mu I \mathbf{S} \qquad \text{weber-meter}$$

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* Note that the magnetic effect of this circular current can be represented by a hypothetical magnet as shown in Fig. 14.01, *right*. This magnet can be assumed to have concentrated poles N', S' of *pole strength* $\pm m$ separated by a distance b. The pole strength can be defined from a relation like Coulomb's law for small electric charges. Then the potential U can be derived for the magnet by the same method as that used in Sec. 10.13 to

so that the magnetic scalar potential can be written

(14.04) $U =$ amperes

Thus a volume distribution of small circular currents can be represented at a point by a vector **M**, the *magnetic moment per unit volume*, just as a volume distribution of electrostatic dipoles can be represented at a point by a vector **P**, the dielectric polarization per unit volume (Sec. 10.13). This method is discussed in Sec. 14.04.

A circular current of the kind described above, flowing in such a manner that no energy is dissipated, does not exist on a macroscopic scale. Currents in metals that are superconductive at temperatures near absolute zero are perhaps most nearly like these hypothetical currents. However, since the magnetic moments of atoms, nuclei, and electrons have been measured indirectly[2] and since these moments appear to correspond to the calculated moments described above for minute hypothetical circular currents, a theory of magnetism in material media can be devised in terms of the experimental data and the theories of atomic physics. Since the detailed quantum-mechanical theory is beyond the scope of this book[3] only a superficial qualitative discussion of this important and fundamental aspect of magnetism is presented below.

The chief contributions to atomic magnetism are assumed to be derived from the following attributes of the electrons associated with the atoms:

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derive the electrostatic potential V in the neighborhood of an electrostatic dipole. The result is

(14.03) U

so that

(14.04) $m\mathbf{b}$

and

(14.04) U

This is the approach of the older theory of magnetic poles, in which $\mathbf{H} = -\boldsymbol{\nabla}U$ was defined as "the force per unit north pole."[1] But note that, if $\mathbf{H}$ is integrated about a closed path *which links current*, the result is *not* zero because U is multivalued (see p. 316).

1. Some kind of motion of each electron about the positive nucleus; in the simple Bohr theory this is assumed to be circular motion about the nucleus as center.

2. Some kind of motion of the charge of an electron about an axis through the electron; this is known as *electron spin*. These motions are assumed to exist without either a dissipation of energy from the atom or a delivery of energy to the atom from an outside source.

It is conceivable that the motions of electrons in an atom of a particular element might be such that their magnetic effects would nullify each other. Perhaps the simplest case would be two electrons having equal but opposite moments, owing to their
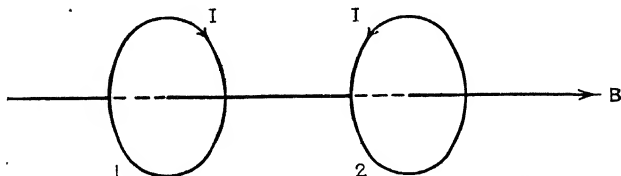
Fig. 14.02.

orbital motions, and equal but opposite spins. In the absence of a magnetic field the atom might be expected to have no magnetic moment attributable to the two external electrons. However, if there were established in the region of the atom a magnetic field parallel to the axis of the several moments, the angular velocities of the circularly moving charges would presumably change. This corresponds to a change in I of Eq. (14.01) and therefore a change in the magnetic moments.

This effect is analogous to the effect of the establishment of a field $\mathbf{B}$ produced at a point far from two equal and opposite circular currents I, Fig. 14.02. Note that the currents are assumed to be equal *before* the field is established by an outside source, so that the field produced at the distant point is zero. While the field is being established, an induced emf is induced in 1, tending to *increase* the current in this coil, while the corresponding emf in 2 tends to *decrease* the current in coil 2. The net result is therefore to produce a magnetic effect like that produced by a small magnet whose axis is parallel to $\mathbf{B}$, with its north pole to the *left* in Fig. 14.02, *i.e.*, a magnetic effect *opposed to the*

applied field. The effect is therefore called the *diamagnetic effect.* In a substance whose atoms have the property described above, the magnetic flux density within the substance is less than the flux density would be in the same volume of vacuum in the presence of the same outside field. Examples of such diamagnetic substances are hydrogen, bismuth, and mercury;[4] their relative permeabilities are less than unity by a few parts in one million.

Problem: A piece of a diamagnetic substance, long compared to the dimensions of its cross section, takes up a position at right angles to a magnetic field in which it is freely suspended. Explain.

If an atom has, in addition to compensating pairs of moments that produce a diamagnetic effect, one or more uncompensated electronic moments of either kind, the atom tends to enhance the applied field. Thus in the absence of 1, Fig. 14.02, the circular current 2 produces a field collinear with **B**. Substances whose atoms have this property are called *paramagnetic substances.* The paramagnetic effect exceeds the diamagnetic effect of compensating pairs of moments. The permeabilities of paramagnetic substances are therefore greater than one.

The most important paramagnetic substances used by electrical engineers are iron, nickel, cobalt, and certain alloys of these elements and of copper, aluminum, molybdenum, and a few other elements as noted in Sec. 14.01. They are characterized by at least one of two properties: (1) permeabilities greatly exceeding one; and (2) the retention of magnetic properties after the removal of the causative magnetic field. The permeabilities of these *ferromagnetic* substances vary with the magnetizing force, and the permeabilities are not single-valued. For example, note the *B-H curve* of an alloy of iron and a small amount of silicon shown in Fig. 14.03. The data are obtained as follows. The sample is a torus, with a toroidal winding through which a controllable direct current flows. A second insulated winding is wound over the primary winding; this secondary winding is connected to a ballistic galvanometer. The arrangement is thus connected as shown in Fig. 14.04. The flux density B and the magnetizing force H are calculated by means of the formulas listed to the right of the figure. The sample is first demagnetized by connecting the primary to a source of alternating current that

is slowly reduced to zero. Presumably this brings the sample to a condition represented in Fig. 14.03 by the origin O. The reversing switch Sw is closed so that a small direct current I is established in the primary winding; the deflection of the bal-

Fig. 14.03.

$$2\pi R = \lambda \qquad R \gg \sqrt{S}$$

$$N_2 \varphi = \int_0^t e\, dt = \int_0^t ri\, dt$$

$$B \approx \frac{rQ}{N_2 S}$$

$$H = \frac{N_1 I}{\lambda}$$
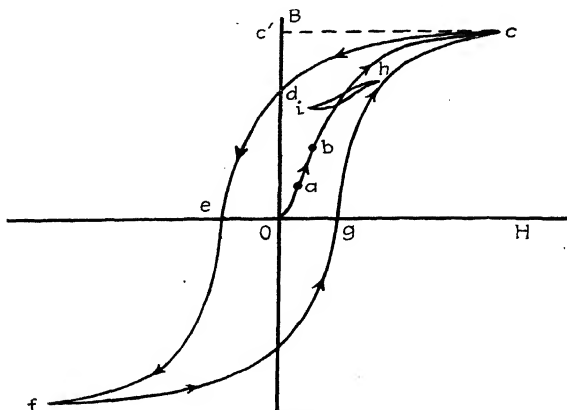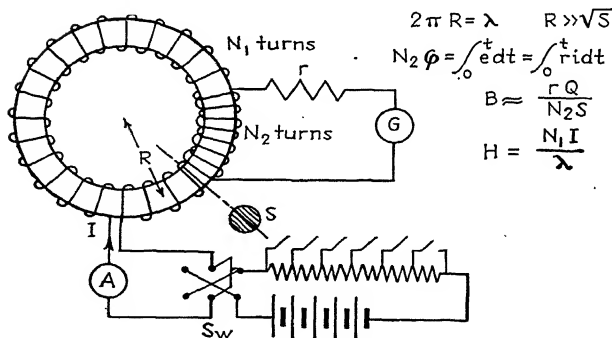
Fig. 14.04.

listic galvanometer G is noted. From these measurements B_a and H_a are calculated, thus specifying the point a. Part of the resistance in the primary circuit is then short-circuited, producing an increase ΔI in I and a ballistic deflection corresponding to the *increase* ΔB in the flux density. Thus H_b corresponding to

$I_a + \Delta I$, and $B = B_a + \Delta B$ can be calculated and represented by point b. The process is continued until B remains *nearly* constant as H is increased further. This condition is loosely called *saturation* (c, Fig. 14.03). When the current is decreased in steps, the flux density B is found to take on values corresponding to the upper curve *cdef*.* Point f, for which B and H are equal but opposite to B and H at c, corresponds to saturation of the core in the sense opposite to that at c. Continuing the process, the curve *fgc* is obtained.†

Note the following commonly used names associated with curves like that shown in Fig. 14.03. The curve *Oabc* is called the *rising B-H curve, or characteristic.* The closed loop *cdefgc* is called a *steady-state hysteresis loop.* The flux density (measured by *Od*, Fig. 14.03) in the sample when H has been decreased to zero is called the *residual magnetism* or *residual flux density.* The ratio of the residual flux density to the maximum flux density (measured by *Oc'*, Fig. 14.03) is called the *retentivity.* The magnitude of the magnetizing force required to produce zero flux density (measured by *Oe*, Fig. 14.03) is called the *coercive force.* These quantities are characteristic of a particular sample at a particular time, and with less accuracy of any sample of a particular kind of ferromagnetic substance at any time.

Actually, the properties of a piece of a ferromagnetic material depend upon its previous magnetic history. Therefore the data of experiments with ferromagnetic materials must be interpreted with care.

A commonly used and relatively unambiguous statement of the B-H characteristic of a ferromagnetic substance, important from the point of view of engineering design, is the *alternating* or *normal B-H* curve. This is the curve for positive B and positive H drawn through the tips (such as c) of steady-state hysteresis loops having different maximum flux densities. These are the curves usually presented in the technical bulletins of the manufacturers of core materials.[5] Further comments on the properties

* The reversing switch Sw is opened to obtain the readings corresponding to point d and then closed in the opposite direction to obtain the readings corresponding to points along *dcf*.

† More precisely, it is *not* obtained. A closed loop such as *cdefgc* is usually obtained only after several cycles of the procedure described above and only if the sample is not subjected to mechanical vibration.

of ferromagnetic materials used in engineering equipment are presented in Sec. 14.02. A more detailed account of the measurement of B-H curves and hysteresis loops may be found in one of the references.[6]

The magnetic permeability μ is the ratio $B/H = \mu$ henrys per meter in the mks system. The permeability of free space is $\mu_0 = 1.257 \times 10^{-6}$ henrys per meter. The permeability μ of a ferromagnetic substance is *not* a constant; it is clear from the example shown in Fig. 14.03 that μ has many values. In many calculations μ may be assumed to be a particular constant without introducing serious error. The following special concepts of permeability are commonly used.

The dimensionless ratio $B/(\mu_0 H) = \mu/\mu_0$ is the *relative permeability;* its value is independent of the system of units.

The *normal permeability* is the ratio $B/(\mu_0 H)$ taken from the *normal* or *alternating magnetization* curve. Curves showing the variation of relative *normal permeability* with H are usually included in manufacturers' technical data.[5]

If a coil of wire on a ferromagnetic core has flowing in it a direct current and a superimposed alternating current, its impedance depends upon (1) the magnitude of the direct current and (2) the magnitude of the superimposed alternating current, as well as on the number of turns and the size, shape, and material of the core. If the a-c component is small compared to the d-c component, the hysteresis loop is similar to that marked *hi* in Fig. 14.03. The *incremental permeability* is the slope of the straight line connecting the tips of such a loop, divided by μ_0. The limit of the incremental permeability as the amplitude of the a-c component approaches zero is the slope of the normal magnetization curve dB/dH divided by μ_0; the quantity $dB/(\mu_0 \, dH)$ is called the *differential permeability.*

It is the purpose of the following paragraph to present a brief outline of the qualitative theory of ferromagnetism, particularly as it is related to the properties represented by the curves of Fig. 14.03.

The magnetic moments of atoms of paramagnetic substances having permeabilities just slightly greater than one are in general small, and there is relatively little tendency for an alignment of a large fraction of these moments in the direction of an applied field. Ferromagnetic materials, on the other hand, have two

properties that contribute to the production of permeabilities far greater than one: (1) the magnetic moments of the atoms are in general high because of several uncompensated moments in each atom; and (2) the crystalline structure of the ferromagnetic substances apparently is responsible for phenomena that cause the moments of a large fraction of the atoms to be aligned in the direction of even a relatively low-intensity applied field. It is thought[7] that interatomic forces* tend to couple together as a single unit a small volume of the substance known as a *magnetic domain*. Means have been devised[7] for examining, under high magnification, the appearance of the surface of ferromagnetic substances while they are being subjected to applied variable fields. It is inferred from these photomicrographic analyses that the magnetic domains of a ferromagnetic substance having a cubic crystal structure† are cubes having edges of the order of 0.01 millimeter long. Since the atomic centers are separated by about 10^{-7} millimeter, each domain would presumably contain of the order of $(10^5)^3 = 10^{15}$ atoms. Since the exchange forces between the atoms of a domain are relatively high, the atoms combine to form a magnetic unit subject to outside influence. On the other hand, the forces between domains are apparently of such form and magnitude that the magnetic moments of the domains have a random distribution in direction when the substance is demagnetized. When an external magnetic field is applied, forces are exerted that tend to align the domains in the direction of the field. A particular domain is assumed to maintain its original direction until, at a particular intensity of the applied field, it suddenly takes up a new direction, more nearly, but not necessarily exactly, in the direction of the applied field. This effect can be "heard." If a thin ferromagnetic sample has wrapped around it a coil of wire connected through a vacuum tube amplifier to a loud-speaker, and if a magnetic field parallel to the axis of the sample is gradually increased, clicks are heard from the loud-speaker. This effect, called the *Barkhausen effect*, is taken as evidence of the sudden alignments of domains within the sample. The initial effect is to align the

* Perhaps interionic forces of electrostatic character; they are called *exchange forces*.

† The form and dimensions of the basic crystal structure of a : can be determined by x-ray diffraction analyses.[8]

domains in an *easy direction* that depends upon the crystal structure and arrangement. If this easy direction happens not to be that of the applied field **H**, further increase in **H** tends to bring about by a continuous process the alignment of the domain moments and **H**. Thus the Barkhausen effect is most pronounced at low and medium flux densities. On the other hand, after the alignment of domain moments and **H** has proceeded as far as possible, the material is said to be saturated. The activities of magnetic domains are supported by energy derived from the source of the applied field and they cause the production of heat in the ferromagnetic substance. Such heat losses, measured in the macroscopic sense in joules per cubic meter per cycle for sinusoidal alternating magnetizing forces, are called *hysteresis losses*. They are discussed in Sec. 14.03. As might be expected, there are associated with the effects described above certain changes in the linear dimensions of a sample of ferromagnetic substance. They are the phenomena of *magneto-striction*.[9] Increase in thermal agitation generally breaks down crystalline structure. Thus at the *Curie temperature* (generally less than 1000°C.) a ferromagnetic substance stops being ferromagnetic, *i.e.*, its relative permeability for higher temperatures does not differ greatly from one.

Because of the complexity of the atomic theory of magnetism, the sections that follow are devoted only to a brief discussion of a few approximate methods for correlating some of the phenomena that are particularly important in the practice of engineering.

14.02. Commonly Used Ferromagnetic Substances.—Metallurgical research has resulted in the production of many different ferromagnetic materials. These materials can be conveniently divided into three classes, although no definite line can be drawn to separate one class from another:

Class A. Properties: Medium (0.3 to 1.0 webers per square meter) flux densities at very low (2 to 200 ampere turns per meter) magnetizing forces, low retentivity, low hysteresis losses. Examples: Permalloy, Mumetal, Hypernik. Uses: Audio-frequency transformers, telephone-system transformers and reactors.

Class B. Properties: High (1.3 to 2.0 webers per square meter) flux densities at medium (200 to 10,000 ampere turns per meter) magnetizing forces, low retentivity, low hysteresis losses. Examples: Electrolytic iron, silicon steels, iron and steel castings.

Uses: Magnetic circuits of power transformers, generators, motors, lifting magnets.

Class C. Properties: Medium (0.3 to 1.2 webers per square meter) flux densities at high (10,000 to 300,000 ampere turns per
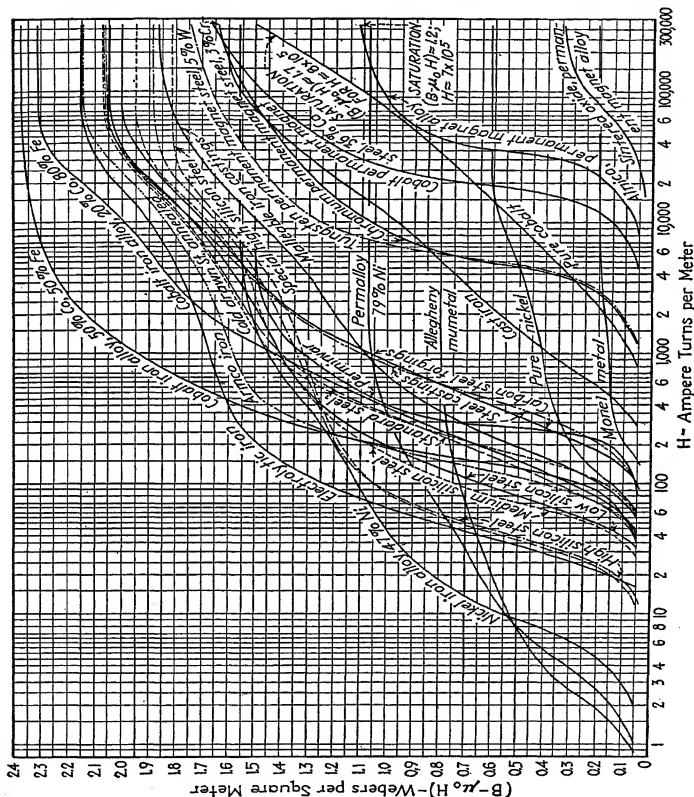
Fig. 14.05.—D-c magnetization curves. (*Courtesy of General Electric Co.*)

meter) magnetizing forces, *high retentivity*. Examples: Cobalt steel, tungsten steel, chromium steel, alnico. Use: permanent magnets.

Examples of rising characteristic magnetization curves are shown in Fig. 14.05. The scales of Fig. 14.05 are marked in

rationalized mks units [$(B-\mu_0 H)$ in webers per square meter and H in ampere turns per meter]. Such units are not now commonly used by design engineers. The following equivalent magnitudes can be used to transfer from one system of units to another. It is unfortunate that the cgs electromagnetic system of units, the English system (for lengths and area), and the mks system are all mixed in the literature of theoretical and applied ferromagnetism.

Magnetizing force:

1 gilbert per centimeter = 0.7958 ampere turns per centimeter
                         = 2.021 ampere turns per inch
                         = 79.58 ampere turns per meter

Magnetic-flux density:

1 kilogauss = 1,000 maxwells (or lines) per square centimeter
            = 6,452 maxwells (or lines) per square inch
            = 0.1   weber per square meter

Although the use of a logarithmic scale for H is a convenient means for comparing ferromagnetic materials differing greatly in magnetic properties, the curves of Fig. 14.05 are of little value as illustrations of the B-H characteristics of a particular substance. The student should plot, *using linear scales for H*, the rising B-H characteristic of one of each of the three classes of ferromagnetic materials discussed above.

14.03. Hysteresis Losses in Ferromagnetic Materials.—Note that the power p, taken from the battery shown in Fig. 14.04 to cause an increase in flux density dB in a time interval dt, is equal to the negative of the product of the emf e induced in the winding N_1

$$(14.05) \qquad e = -N_1 \frac{d\phi}{dt} = -N_1 S \frac{dB}{dt} \qquad \text{volts}$$

and the current i in this winding

$$(14.06) \qquad i = \frac{\lambda H}{N_1} \qquad \text{amperes}$$

The notation is that used in Fig. 14.04. Therefore

$$(14.07) \qquad p = \frac{dw}{dt} = \lambda S H \frac{dB}{dt} \qquad \text{watts}$$

and the increase in energy *dw per unit volume* is

(14.08) $$dw = H \, dB \qquad \text{joules per cubic meter}$$

Therefore, the establishment of a magnetic field in a region where the flux density is initially zero requires an expenditure of energy *w* by the source of the field

(14.09) $$w = \int_0^B H \, dB \qquad \text{joules per cubic meter}$$

Problem: Prove that for a ring sample such as that shown in Fig. 14.04, the *total energy W* required to establish a flux ϕ when the flux is initially zero is

(14.10) $$W = \int_0^\phi \Re\phi \, d\phi \qquad \text{joules}$$

and that this reduces to

(14.11) $$W = \tfrac{1}{2}\Re\phi^2 \qquad \text{joule}$$

when the permeability of the core remains constant during the establishment of the flux ϕ.

Now if the material of the core is ferromagnetic, the integral of $H \, dB$ from a maximum value B_m to a minimum value $-B_m$ and back to B_m is *not* zero. For example the integral $H \, dB$ for the loop *cdefgc* of Fig. 14.03 is proportional to the area of the loop. Thus there is a loss of energy from a source of sinusoidal emf of frequency f cycles per second if such a source is connected to the coil N_1, Fig. 14.04, even though N_2 is disconnected. If the area of the hysteresis loop obtained under these circumstances is A, the energy loss w is

(14.12) $$w = KA \qquad \text{joules per cubic meter per cycle}$$

in which K is the factor of proportionality. If the loop is "traversed" f times per second the average power loss P from the circuit is

(14.13) $$P = fKA \qquad \text{watts per cubic meter}$$

Steinmetz[10] studied the relation between the area of the hysteresis loop and the maximum flux density B_m for commonly used ferromagnetic materials. He arrived at an empirical relation which, though approximate, is the basis of calculation of hysteresis losses for the cores of a-c power equipment such as alternators, motors, and transformers. Expressed in the terms used above,

his finding is that

$$(14.14) \qquad KA \approx \qquad \text{joules per cubic meter per cycle}$$

in which η' is called the *Steinmetz coefficient of hysteresis.* *   Thus, subject to all the qualifications and limitations listed above, the hysteresis loss in a core of volume V undergoing cyclic changes of flux density of maximum value B_m produced by a sinusoidal emf of frequency f applied to a coil on the core is approximately

$$(14.15) \qquad P \approx \eta' V f B_m^{1.6} \qquad \text{watts}$$

This power is dissipated as heat by the core.   Note, however, that the core itself is an electrical conductor; the changing flux in it produces emfs that in turn cause *eddy currents* to flow in the core.   These currents also produce heat—the *eddy-current loss*—so that the total core loss comprises the hysteresis loss, Eq. (14.15), and the eddy-current loss, which is discussed in Sec. 14.05   The Steinmetz coefficient of hysteresis *in cgsem units* ranges from 0.00015 for alloys such as the permalloys to nearly 0.1 for permanent magnet steels.

Since the *B-H* curves of ferromagnetic substances are not predictable from the theoretical analysis, the theory of a-c machinery in which such substances are used is usually developed in one of three ways.

1. The permeability of the ferromagnetic material is assumed to be constant.   This is clearly an incorrect assumption.   The results of such developments are often grossly in error.

2. An attempt is made to represent the normal *B-H* curve by means of an algebraic function $B = f(H)$.

3. The experimental *B-H* curve is used as the basis of a numerical integration of the differential equations of the particular problem, or this curve is used on the differential analyzer, which solves the problem mechanically.

The second procedure has been developed by Fröhlich, among others.   He found[11] that the normal *B-H* curves of many ferro-

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* The symbol η' is used here because the symbol η was used by Steinmetz for the coefficient in the cgsem system.   In this system the hysteresis loss per cycle is $\eta B^{1.6}$ ergs per cubic centimeter; B is expressed in gauss.   The coefficient η' of Eq. (14.14) is $2.512 \times 10^5 \eta$; values of η are listed in handbooks[14] and in the technical literature of companies manufacturing ferromagnetic materials.[5]

magnetic substances could be sufficiently accurately represented
by a hyperbola

(14.16)* $B = \dfrac{aH}{b + H}$ webers per square meter

14.04. Magnetic Moment per Unit Volume or Intensity of Magnetization.
As noted in the beginning of Sec. 14.01, it is possible to develop a theory
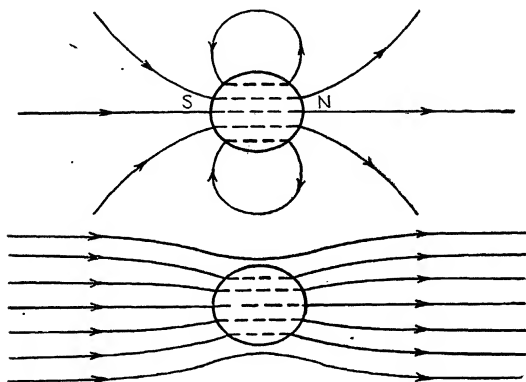
Fig. 14.06.

of paramagnetic phenomena by assuming that the amperian currents in
the paramagnetic substance can be represented by the *magnetic moment per
unit volume* or *intensity of magnetization.* This development, discussed
briefly below, follows closely the development of the theory of linear iso-
tropic dielectrics discussed in Sec. 10.13. Unfortunately, the ferromagnetic
substances to which this theory would presumably be applied most usefully
are neither linear nor isotropic. *This discussion is presented chiefly, therefore,
to acquaint the student with the classical approach to magnetic theory.*

A more comprehensive analysis, leading to the results stated in this section
and derived from the concept of the vector potential, is given in Appendix E.

The upper diagram of Fig. 14.06 represents a uniform field *B.* Below
this is a representation of the field produced by a spherical magnet. Suppose
that the paramagnetic sphere is caused to be magnetized by the formerly
uniform field. The resultant field is represented by the bottom diagram of

* **Problem:** Calculate *a* and *b* for the Fröhlich functions for one of each
of the three classes of ferromagnetic materials described in Sec. 14.02.

Fig. 14.06. Qualitatively the result is easy to describe. The amperian currents of the paramagnetic sphere are initially oriented in a random manner; they produce no poles. When the sphere is placed in the uniform field there is a tendency for the alignment of the magnetic moments in the direction of the field, as described in Sec. 14.01. The result is a concentration of the lines representing B in the sphere. Quantitatively, the phenomena can only be correlated in a simple manner if:

1. The magnetic moment per unit volume or intensity of magnetization $\mathbf{M}$ (see Sec. 14.01) has the same direction as the magnetizing force $\mathbf{H}$ and,

2. The magnetic moment per unit volume $\mathbf{M}$ is proportional to $\mathbf{H}$.

Under these circumstances, with the additional qualification that $\mathbf{H}$ and $\mathbf{B}$ are collinear, the flux density within the sphere can be expressed in the form

(14.17) $$\mathbf{B} = \mu_0\mathbf{H} + \mathbf{M}$$ webers per square meter

(14.18) $$\mathbf{M} = \mathbf{B} - \mu_0\mathbf{H}$$ webers per square meter

(14.19) $$\mathbf{M} = \left(\frac{\mu}{\mu_0} - 1\right)\mu_0\mathbf{H} = \chi_m\mu_0\mathbf{H}$$

(14.20) $$\frac{\mu}{\mu_0} = \chi_m + 1$$ (numeric)

These relations correspond to those derived from the classical theory of a linear, isotropic, polarized dielectric (Sec. 10.13). The numeric, χ_m, is called the *magnetic susceptibility*.

Note that, since $\nabla \cdot \mathbf{B} = 0$,

(14.21) $$\nabla \cdot \mu_0\mathbf{H} = -\nabla \cdot \mathbf{M}$$

Within a homogeneous isotropic paramagnetic material these quantities are both zero. If they were not, some kind of "magnetic charge" density could be defined as being equal either to $\nabla \cdot \mu_0\mathbf{H}$ or to $-\nabla \cdot \mathbf{M}$. Such effects are found only at surfaces between two substances of different magnetic properties. At such surfaces the values of $\mathbf{M}$ and $\mathbf{H}$ change. The divergence of $\mathbf{M}$ at such a surface is

(14.22)* $$\mathbf{M}_1 \cdot d\mathbf{S} - \mathbf{M}_2 \cdot d\mathbf{S} = dm$$ webers

i.e., the difference between the normal components of the intensity of magnetization on the two sides of the surface. The quantity dm so defined is called the *magnetic pole strength*. It is analogous to the surface charge density of polarization of the surface between two dielectrics.

At the bounding surface between two paramagnetic substances, there is presumably a refraction of the lines representing $\mathbf{B}$, just as there is refraction of the lines representing the dielectric flux density $\mathbf{D}$ at a bounding surface between two different homogeneous isotropic dielectrics. Thus, in Fig. 14.07 the normal components of $\mathbf{B}$ are equal ($\nabla \cdot \mathbf{B} = 0$) so that

(14.23) $$\cos\theta_1 \qquad \cos\theta_2$$

---

* According to the definitions used in this book, the force between two magnetic poles is $F = mm'/4\pi\mu_0 r^2$ newtons; the dimensions of m given above are in accord with this result; H is then a vector having the dimensions force per unit north pole ($+m$)

or

(14.23a) $: \cos \theta_2$

Since the line integral of **H** around a closed loop perpendicular to the surface does not link any current

(14.24) $$\oint \mathbf{H} \cdot d\mathbf{l} = H_1 \sin \theta_1 - H_2 \sin \quad : 0$$

It follows that

(14.25) $$\frac{\tan}{\tan \theta_2}$$

This simple theory, attractive because it is analogous to the classical theory of polarization in dielectrics, has little practical significance to engineers because

1. In ferromagnetic materials the intensity of magnetization is seldom proportional to the magnetizing force.

2. The intensity of magnetization in ferromagnetic substances is often not collinear with **B** and **H**.

3. Ferromagnetic substances are neither homogeneous nor isotropic.

It is clearly important to know just what assumptions can be made safely in each engineering problem in magnetism and then to choose the simplest applicable theoretical approach.
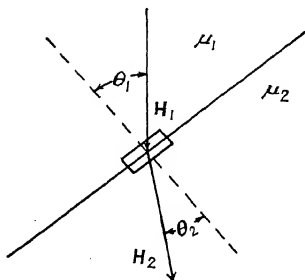
Fig. 14.07.

14.05. Eddy Currents.—As noted in Sec. 14.03, the changing flux in the ferromagnetic core of a machine causes induced currents that in turn produce heat. This eddy-current loss is minimized by using ferromagnetic alloys of high resistivity in the form of thin sheets or *laminations*. Although a detailed study of eddy-current losses involves the solution of differential equations similar to Eq. (13.33), a simpler approximate analysis is instructive.

The rectangle in Fig. 14.08 represents the cross section of a lamination of ferromagnetic material of resistivity ρ ohm-meters. Suppose that the length of this lamination (into the paper) is 1 meter and that the flux density over the cross section is uniform and varies sinusoidally with time. Assume that $b << a$ so that there need be considered only the flux within the rectangle a meters long and x meters wide, which is

(14.26) $\phi = axB = 2axB_m \sin \omega t$ webers

and the emf e induced in the periphery of the rectangle is

$$(14.27) \qquad e = -\frac{d\phi}{dt} = -2\omega a x B_m \cos \omega t \qquad \text{volts}$$

The conductance G of the hollow rectangular form of length 1 meter and thickness dx meters is

$$(14.28) \qquad \overline{(2a}$$

and since the dimension b of the laminations commonly used is negligibly small compared to the dimension a, the following approximation is accurate enough:
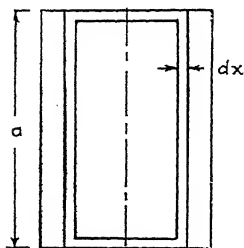
$$(14.29) \qquad G = \frac{dx}{2a\rho}$$

The average power dissipation in this particular "cylinder" is

$$(14.30) \quad dP' = f \int_0^{1/f} Ge^2 \, dt =$$

$$\int_0^1 \qquad \cos^2 2\pi f t \, dt$$

The average power dissipated throughout the lamination is

$$(14.31) \qquad P' = \frac{\omega^2 B_m^2}{3\rho} a \frac{b^3}{8} \qquad \text{watts}$$

To express the power in watts per cubic meter, to correspond to Eq. (14.13), P' is divided by ab, the volume of the lamination.

$$(14.32) \qquad P = \frac{}{6\rho} \cdot B_m^2 \qquad \text{watts per cubic meter}$$

The factor $\pi^2/6\rho$ is often taken as the eddy-current coefficient ϵ'. Thus combining Eqs. (14.15) and (14.32), the total core loss is

$$(14.33) \qquad P = \eta' f B_m^{1.6} + \epsilon' b^2 f^2 B_m^2 \qquad \text{watts per cubic meter}$$

Note that the derivation of Eq. (14.32) takes no account of the fact that the eddy currents themselves produce flux in the iron. At frequencies below 100 cycles per second the error due to this omission is practically negligible.

14.06. The Force on a Current-carrying Conductor in a Magnetic Field.—Two current-carrying conductors exert mutual forces on each other, as noted in Chap. XIII. Ampère measured the forces for a few special cases and he postulated certain general relations among the forces, currents, and space coordinates of the conductors. These relations are in accord with subsequent experiments, with the possible exception of cases involving charges moving at extremely high velocities ($> 10^7$ meter per second). These relations can be expressed vectorially[12] in terms of differential lengths of each of the two circuits, or the forces
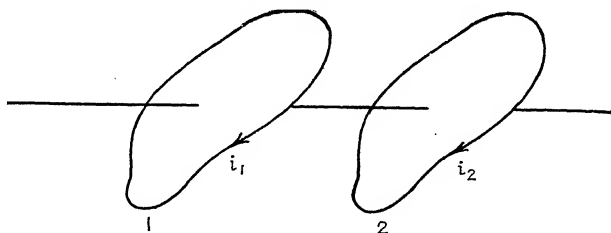
FIG. 14.09.

can be expressed for the whole circuits. The latter procedure is discussed below for a few special cases involving conductors in regions for which $\mu = \mu_0$.

If the currents i_1 and i_2 in circuits 1 and 2, Fig. 14.09, are initially zero and the circuits are fixed with respect to each other, the sources of the currents supply energy to produce the self- and mutual-magnetic effects associated with the circuits and the region in which they are fixed, as the currents are increased from 0 to I_1 and I_2. The self-induced emfs

$$(14.34) \qquad e_{11} = -\frac{d\phi_{11}}{dt} \qquad \text{and} \qquad \overline{\qquad dt \qquad}$$

can be considered separately from the mutually induced emfs

$$(14.35) \qquad e_{12} = -\overline{\qquad dt \qquad} \qquad \text{and} \qquad -\frac{d\phi_{21}}{dt}$$

because the system is linear and the principle of superposition can therefore be applied. According to this notation, the sources of the currents i_1 and i_2 must supply energy $-\Sigma ei\, dt$ joules

to produce the change in fluxes $\sum \dfrac{d\phi}{dt}$. Thus the total energy dW in time dt is

$$(14.36) \qquad dW = e_{11}i_1\,dt + e_{22}i_2\,dt + e_{12}i_1\,dt + e_{21}i_2\,dt \qquad \text{joules}$$

Substituting from Eqs. (14.34) and (14.35), Eq. (14.36) becomes

$$(14.37) \qquad dW = i_1 d\phi_{11} + i_2 d\phi_{22} + i_1 d\phi_{12} + i_2 d\phi_{21} \qquad \text{joules}$$

But the $d\phi$'s can in this case be specified in terms of the self-inductances, mutual inductances, and currents

$$(14.38) \qquad \begin{aligned} d\phi_{11} &= L_1\,di_1 & d\phi_{12} &= M\,di_2 \\ d\phi_{22} &= L_2\,di_2 & d\phi_{21} &= M\,di_1 \end{aligned}$$

Thus the total energy W is

$$(14.39)^* \quad W = \int_0^{I_1} L_1 i_1\,di_1 + \int_0^{I_2} L_2 i_2\,di_2 \pm \int_0^{I_1 I_2} M\,d(i_1 i_2)$$

if it be assumed, as written in Eq. (14.38), that $M_{12} = M_{21}$ (see below). Integrating Eq. (14.39)

$$(14.40) \qquad W = \tfrac{1}{2} L_1 I_1^2 + \tfrac{1}{2} L_2 I_2^2 \pm M I_1 I_2 \qquad \text{joules}$$

Note the similarity of Eqs. (14.40) and (11.22) for the energy of the electrostatic field of two charged conductors.

Now if, after the currents I_1, I_2 are established, the conductor 1 is imagined to be displaced a distance $\mathbf{ds}$, the force $\mathbf{F}_s$ exerted on this conductor in the direction of $\mathbf{ds}$ can be calculated from

$$(14.41) \qquad \mathbf{F}_s = \frac{\partial W}{\partial s}\,\mathbf{s}_1 \qquad \text{newtons}$$

where $\mathbf{s}_1$ is a unit vector in the direction of $\mathbf{ds}$. In Eq. (14.40), only M is a function of $\mathbf{s}$; the other quantities are constant with respect to s. Therefore

$$(14.42) \qquad \mathbf{F}_s = \pm I_1 I_2 \frac{\partial M}{\partial s}\,\mathbf{s}_1 \qquad \text{newtons in the direction } s$$

If $\mathbf{ds}$ is in such a direction as to cause the mutual inductance to increase, and if both I_1 and I_2 link the mutual flux in the right-

* The third term of the right-hand side of Eq. (14.39) is written with a plus sign if the mutual flux is linked by both currents in the right-hand-screw direction; the negative sign is used if one current links the mutual flux in the right-hand-screw direction and the other current links the mutual flux in the left-hand-screw direction.

hand screw direction, the force is in this case positive, *i.e.*, the circuit 1 would move in the direction **ds** if it were free to move in that direction. If **ds** is so chosen that it is in the direction in which the conductor 1 would move if it were free to move in *any* direction, then $\mathbf{F}_s$ is the total force on conductor 1. Since experiment shows that there is an equal but opposite force exerted on conductor 2, it follows that the two mutual inductances

$$(14.43) \qquad M_{12} = \frac{d\phi_{12}}{di_2} \qquad \text{and} \qquad M_{21} = \frac{d\phi_{21}}{di_1}$$

have the same derivatives with respect to *s*. Since this relation is true for *all* values of *s*, and since for *s* infinite M_{12} and M_{21} are both zero, it follows that

$$(14.44) \qquad\qquad\qquad = M_2$$

as it was assumed above [see Eq. (14.38)].

Note that the sign of $\mathbf{F}_s = \pm I_1 I_2 \frac{\partial M}{\partial s} \mathbf{s}_1$ depends not only upon the increase or decrease of M for an assumed displacement ds, but also upon the relative directions of $I_1 I_2$. If the flux through conductor 1 produced by I_1 is in the same direction as the flux through conductor 1 produced by I_2, the positive sign is used. On the other hand, if I_1 and I_2 link the mutual flux in opposite directions, the negative sign is used. If follows that two parallel conductors carrying currents in the *same* direction attract each other, while two parallel conductors carrying current in the opposite directions repel each other. There may be superimposed upon these forces additional forces of attraction or repulsion due to the electrostatic effects of the potentials of the conductors (see Sec. 11.06).

14.07. The Forces on a Moving Electric Charge.—Forces on electric charges have been classified, in the developments presented in this book, as

1. Electropositional or electrostatic forces, depending upon the magnitudes and positions of electric charges. These are discussed in Chap. I in relation to electric circuits and in Chaps. X and XI for charges in electrostatic fields.

2. Electromotional forces, depending upon the magnitudes, positions, velocities, and accelerations of electric charges. These are discussed in Chap. I for electric circuits and in Chap. XII

in relation to induced emfs in conductors and the electromotion-field intensity $\mathbf{E}''$, which has the dimensions of force per unit charge.

3. Electrofrictional forces, which oppose the motion of electric charges through material media.

4. Electrochemical forces, which maintain a separation of unlike charges at surfaces bounded by two different materials, such as the surface between a metal and an electrolyte.

Another force, which is exerted on moving charges, depends upon the magnitude and velocity of the moving charge and upon the magnitude of the magnetic-flux density in the neighborhood of the charge. *This force is perpendicular to the direction of motion of the charge and to the direction of the magnetic flux density.*

A convenient method for symbolizing this latter force in such a manner that it is perpendicular to both the velocity of the charge and the flux density is to represent it by the cross product of two vectors. Thus it is consistent with experiment to express this force $\mathbf{F}$ by the relation

$$(14.45) \qquad\qquad \mathbf{F} = Q\mathbf{u} \times \mathbf{B} \qquad\qquad \text{newtons}$$

in which Q is the charge which moves with a velocity $\mathbf{u}$ in a magnetic field of flux density $\mathbf{B}$.

The magnitude of the force $\mathbf{F}$ of Eq. (14.45) is the electromagnetic component of the force that acts, for example, on the conductors of a two-wire transmission line; it is a force in the plane of the conductors, tending to push them apart. There is usually, in addition, an *electrostatic* force of attraction between the conductors. In certain electronic devices, electrons and positive ions move at high velocities in a vacuum. Under these conditions, the force $\mathbf{F}$ of Eq. (14.45) may be the only force that needs to be considered for determining the trajectory of the moving charges. Examples of the use of Eq. (14.45) for the calculation of such electronic and ionic trajectories are worked out in Chap. XV.

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Problems

14.01. The currents in two air-core coils of self-inductance 4 and 10 mh. are 10 and 25 amp. When the currents are in such direction that the magnetic energy in the system is maximum, that energy is 4.26 joules. Calculate the mutual inductance between the coils and the magnetic energy if the current in one coil is reversed.

14.02. Two coils having the same self-inductance are connected in series. When a voltage of 110 volts (d.c.) is applied to the terminals of the combination, the energy stored in their field is W joules; if the connections of *one* coil are interchanged and 55 volts (d.c.) applied to the combination, the stored energy is W joules. Calculate the coefficient of coupling ($k = M/\sqrt{L_1 L_2}$) between the coils.

14.03. A coil ($L_1 = 0.5$ h, $R_1 = 110$ ohms) is coupled to a second coil ($L_2 = 0.1$ h, $R_2 = 22$ ohms); the coefficient of coupling is 0.6. If a voltage of 110 volts d.c. is connected to the first coil at $t = 0$; (*a*) what is the open-circuit voltage of the second coil at $t = 0.01$ sec.; (*b*) how much charge would pass through coil 2 if it were initially short-circuited?

14.04. Calculate the force on either coil of Problem 13.10.

14.05. A current transformer has a toroidal core having a square cross section with sides $\sqrt{2}$ in. long and a mean length of 20 in. The secondary winding comprises 500 turns uniformly wound on the toroid; the total secondary resistance (including the meter) is 2 ohms. The primary is a single cylindrical conductor through the center of the toroid, perpendicular to its plane; it carries a current $\sqrt{2}\,2,000 \sin 2\pi 60t$. Assuming that the

relative permeability of the core is 2,000 and that there is no secondary leakage, calculate the current in the secondary circuit, including its phase with respect to the primary current.

14.06. A short-circuited coil of resistance R_1 and self-inductance L_1 is coupled to a second coil by a mutual inductance M. If the second coil is held stationary and if the current in it is $I_2 \sin \omega t$, calculate the average force exerted on the first coil.

14.07. A uniform toroidal coil of 600 turns is wound on a laminated core of transformer steel of rectangular cross section, 0.014 in. thick. The inner radius of the core is 15 cm., the outer radius is 17 cm., and the width of the core perpendicular to these radii is 5 cm. A direct current of 0.7 amp. flows in the coil. Calculate: (*a*) the magnetomotive force for any radius

$$x \ (15 \leq x \leq 17 \ \text{cm.})$$

(*b*) the magnetizing force at any radius x, within the same limits; (*c*) the average magnetizing force; (*d*) the magnetizing force at the mean radius $x_m = 16$ cm.; (*e*) the total flux through the ring; (*f*) the average flux density in the ring; (*g*) the average relative permeability of the core; (*h*) the "reluctance" taken as the ratio of mmf to flux; (*i*) the "reluctance" taken as the ratio of the mean length of the core to the product of μ (the average permeability) and the cross section of the core; (*j*) the quantities listed in (*a*) to (*i*) if the iron core were replaced by a wood core; (*k*) the current required to produce the flux calculated in (*e*) if the core were made of wood; (*l*) the hysteresis and eddy-current losses for the iron-core toroid for an applied voltage of 110 volts, 60 cycles; (*m*) for 110 volts, 25 cycles; (*n*) the inductance for the conditions of (*l*); (*o*) the inductance if the current through the coil were 0.7 amp. d.c. and the a-c applied voltage 1 volt, 60 cycles; (*p*) impedance for (*o*) and for an applied a-c voltage of 1 volt, 180 cycles; (*q*) the inductance of the coil with wood core; (*r*) the inductance for an applied voltage of 110 volts, 60 cycles, if there is an air gap of 0.5 mm.; (*s*) 1 mm.; (*t*) 2 mm.; (*u*) 4 mm.

14.08. A toroid having a mean length of 10π in. and a square cross section of 1 sq. in. consists of three equal sections, each of mean length $10\pi/3$ in.; the sections are made of cast iron, soft steel casting, and annealed sheet steel. A current of 2 amp. flows through a uniform 500-turn winding on the toroidal core. Calculate the mean flux density, the magnetizing force, and intensity of magnetization in each section, the pole strengths at the boundaries between sections, the reluctance of each section, and the ratio of the total flux in the core to the current through the coil. Is the last calculated number the self-inductance of the coil?
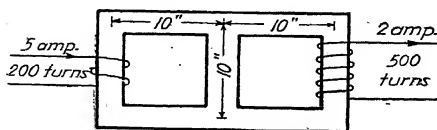
PROB. 14.09.

14.09. The cross section of each leg of the transformer-steel core shown above is 1 sq. in. Calculate the flux in each leg of the core, assuming that there is no leakage flux. If a 500-turn coil were wound on the middle leg of the core, and the current in it varied from 0 to 5 amp. d.c., how would the self-inductance of the coil on the left leg, for small a-c voltages, vary?

14.10. A cylindrical steel bar with hemispherical ends is magnetized so that the pole strengths are $\pm m$. The total length of the bar is $d + 2r$, the radius of the hemispherical ends being r. Calculate the magnetic potential and the magnetizing force at a point distant a from one end of the magnet on the extended axis of the bar.

14.11. A sample of iron is found to have a relative permeability μ/μ_0 of 620 at a flux density of 1.2 webers per sq. m. A bar of this material 1 m. long and 4 cm. in diameter is wound uniformly with 2,000 turns of insulated wire. What current is required to produce a flux density of 1.2 webers per sq. m. at the center of the solenoid? What current would be required to produce the same flux density if the iron core were removed?

14.12. The flux ϕ in the core of a transformer is related approximately to the current I in a 500-turn winding on the core by

$$\frac{0.03I}{2 + I} \quad \text{webers}$$

Calculate the increase in energy per unit volume of the core when the current changes from 1 to 2 amp.

14.13. The ironclad electromagnet shown above exerts an upward force of 100 lb. when in the position shown. The iron core is made up of sheet-steel laminations, 0.05 in. thick. The cross sections of the iron yoke and plunger are each 3 sq. in. What would be the total work done in pulling the plunger down an additional $\frac{1}{2}$-in.? Neglect eddy currents and hysteresis losses.
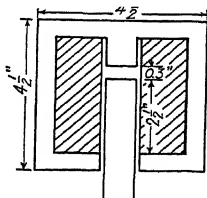
PROB. 14.13.

14.14. The ironclad electromagnet shown below is required to lift a plunger

PROB. 14.14.

which weighs 12 lb. from a position 1 in. distant from the brass stop. When the current is interrupted, the plunger must return to its original position owing to its own weight. The cross section of the stop and plunger is 2 sq. in. and of the yoke 1 sq. in. A brass disk of 0.05 in. thickness attached to the iron stop limits the travel of the plunger. If the residual flux is 10 per cent of the maximum flux, will the relay work? If not, how can it be made to work?

14.15. The total core-loss measurement for a 50 hp., two-pole d-c motor provides these data:

Input current, amp.	Rpm	Total core loss, watts
50	600	1127
50	700	1403

The core has a volume of 6 cu. ft., and is constructed of silicon-steel laminations 0.06 in. thick.   The maximum flux density is 85 kilolines per square inch.

a. Find the hysteresis loss at 600 and 700 rpm.

b. Find the eddy-current losses at 600 and 700 rpm.

c. What are the eddy-current and hysteresis coefficients?

14.16. A d-c relay is constructed as shown below (in two views).   The core has a cross section of 0.25 sq. in. and a mean length of 8 in.; the per-

PROB. 14.16.

meability of the core material is 2,000 cgs units.   The brass shims are 0.06 in. thick. When no current flows in the winding each air gap is 0.10 in. in length.
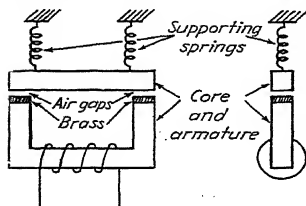
a. If a current of 0.5 amp. flows in the coil (500 turns) and if the armature is blocked, what is the downward force on the armature, produced by the current in the coil?

b. If the armature is permitted to come in contact with the brass shims while the current is flowing, what is the force on the armature?

c. What is the difference in the energies stored in the field in a and b?

14.17. A round steel ring has a mean diameter of 8 in. and a cross-sectional area of 2 sq. in.   It is made up of annealed sheet-steel rings 0.0250 in. thick. The ring is uniformly wound with 2,010 turns of No. 30 A.W.G. wire. When a 60-cycle current of 0.0625 amp. flows through the winding, the total power loss as read by a wattmeter is 53.7 watts.   When the current is increased to 0.25 amp., the wattmeter reading increases to 104.2 watts.

a. What is the hysteresis loss in each case?

b. What is the eddy-current loss in each case?

14.18. A coil having 1,000 turns of No. 24 A.W.G. copper is wound on a closed core having a mean length of 15 in. and a cross section of 1 sq. in. The mean length of turn is 4.4 in.   The core is made up of 0.014 in.-thick laminations of standard annealed sheet steel.   A 60-cycle sinusoidal alternating current having an effective value of 0.5 amp. flows in the coil.   How much power in watts is dissipated as heat in this unit due to its (a) resistance? (b) hysteresis?   (c) eddy currents?

14.19. A beam of electrons is projected into a uniform magnetic field $B = 10^{-4}$ weber per sq. cm. at an angle of 60 deg. with respect to the vector **B.**   The velocity of the electrons as they enter the field is 10^5 m. per sec. What is the path of the beam in the magnetic field?

14.20. A beam of hydrogen ions enters a uniform magnetic field at right angles to the field.   What must be the magnitude of the flux density in order that the ions traverse a complete circle in 10^{-5} sec.?

CHAPTER XV

THE THERMIONIC EFFECT AND ELECTRON BALLISTICS

15.01. Introduction.—The phenomena underlying the design and uses of vacuum tubes, cathode-ray tubes, x-ray tubes, photoelectric tubes, the cyclotron, and other *electronic or ionic* devices are the subjects of theories and design methods that are becoming increasingly important in electrical engineering. These subjects are now presented to students in special courses with titles such as *High-frequency Alternating Currents* or *Electronics*. It is the purpose of this and of the next chapter to point to a few of the important bases upon which the material of such courses is developed.

Note that in this book 14 chapters have been devoted to elementary electric-circuit analysis and field theory, while one is devoted to electronics and one to the elements of electromagnetic radiation. Since telegraphy, telephony, radio, television, and electronic control of industrial processes now comprise a large and increasing fraction of the field of electrical engineering, this apparently lopsided organization requires an explanation.

First, the material of the first 14 chapters is just as important to the "communications engineer" as it is to the "power engineer." Thus the emphasis given to this material by the allotting of so large a fraction of this book to it is justified because all electrical engineers use electric-circuit theories and the theories of electrostatic and electromagnetic fields.

Second, as noted several times in earlier chapters, a general theory of electromagnetism based upon experiments involving atoms and subatomic entities such as electrons and protons is not adaptable for use in an introductory course for electrical engineering students. Such a course would be, at least at present, logical but unteachable. Third, during this transition stage when knowledge of and need for training in electronics and radiation are increasing so rapidly, these subjects are treated in special courses, usually presented to seniors.

For these reasons the material that follows is chosen chiefly for the purpose of introducing the student to subjects he will study in detail later.

15.02. The Thermionic Effect.*—It has been emphasized throughout this book that the conduction electrons in metals are normally confined within the boundaries of the metal. A qualitative discussion of the *potential-barrier* theory of such boundaries is given in Secs. 3.01 and 3.09.

Electrons may acquire sufficient energy to overcome the potential barrier and thus be ejected from the surface of a metal in several ways, of which the following are examples.

1. *Thermionic emission:* some of the free electrons may derive sufficient energy from the effects of the metal's being heated.

2. *Cold emission:* sufficiently intense potential gradients near the surface of a metal in a vacuum may "pull" electrons from the surface, even though its temperature is too low for appreciable thermionic emission.

3. *Secondary emission:* high-velocity electrons may impinge on a surface in such a way that their energies are partly transferred to electrons in the surface, causing the latter electrons to be emitted.

4. *Photoelectric emission:* the energy of a beam of radiation—infrared, visible, ultraviolet, x-ray, or γ-ray—may be transferred to electrons in a metal, causing them to be ejected from the surface as *photoelectrons.*

It is the purpose of this section to discuss briefly the phenomena of thermionic emission.

According to the kinetic theory of matter, the free electrons in a metal acquire energies whose most probable magnitude depends upon the temperature of the metal.† The most probable energy can be expressed as $\frac{1}{2}m\bar{v}^2$ joules, in which $\bar{v}$ is the *rms velocity,* also called the *most probable velocity;* as kT joules, in which k is the Boltzmann constant (1.37×10^{-23} joule per electron per degree Kelvin) and T is the absolute (Kelvin) temperature; or as V_T electron volts, *i.e.,* the difference of poten-

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* For a more detailed discussion see *Fundamentals of Engineering Electronics;* W. G. Dow; John Wiley & Sons, Inc., New York, 1937, from which specific references are listed at the end of this chapter.

† Their motions in the conductor of a resistor produce a *noise voltage* whose *rms* value E is $E = \sqrt{4RkT(\Delta f}$; k is the Boltzmann constant, R the resistance in ohms, T the absolute temperature, and Δf the frequency band width.

tial through which the electron must fall to acquire the velocity $\bar{v}$. In order to analyze the phenomena of thermionic emission from a surface, the component of $\bar{v}$ that is perpendicular to the surface is the relevant velocity because it is this component that contributes to the determination of the probability of escape of the electron. Thus the phrase *directed energy* is often given to a quantity $\frac{1}{2}m\bar{v}^2$, to imply that it can do work along the line representing $\bar{v}$ vectorially.

The potential barrier at the surface of a metal can be usefully described in electron-volts V_w; the energy an electron would acquire if it fell through this difference of potential is the energy that is required of the electron to overcome the barrier and to escape from the surface. The quantity V_w depends upon the material of the emitting surface; it is called the *work function.*

Even if the work function can be measured or calculated from corollary experimental data, the relation among the rate of emission of electrons J from a flat surface, the temperature T, and the constants associated with the particular emitting surface is not easy to calculate. The chief difficulty lies in the determination of the statistical distribution of energies among the electrons. This theoretical approach to the problem is discussed in one of the references[8] listed at the end of this chapter. The experimentally determined values of thermionic emission for commonly used materials are in accord in most important respects with the theoretically determined values. The thermionic emission J amperes per square meter can be calculated from

$$(15.01) \qquad J = AT^2\varepsilon^{-\frac{V_w}{V_T}} \quad \text{amperes per square meter}$$

where J = total thermionic emission per unit area of the emitting surface.

  A = a constant characteristic of the material of the emitter.

  T = the temperature of the emitter, degrees Kelvin.

  V_w = the work function of the material of the emitter, electron-volts.

  V_T = mean electron energy in electron-volts of the electrons in the emitter at temperature T. Since $eV_T = kT$,

$$V_T = \left[\frac{(1.37 \times 10^{-23})}{(1.59 \times 10^{-19})}\right] T = 0.86 \times 10^{-4}T,$$

k being the Boltzmann constant, which is 1.371×10^{-23} joule per degree absolute.

It follows that V_w and the constant associated with V_T can be combined into a constant b, so that Eq. (15.01) can also be written

(15.02) $$J = AT^2\varepsilon^{-\frac{b}{T}} \quad \text{amperes per square meter}$$

It should be noted that $V_T < V_w$ for known emitting surfaces and practicable temperatures. Thus electrons having energy of just V_T electron-volts are not likely to be emitted. This energy is, however, just the most probable electron energy; at the temperature corresponding to V_T some electrons have energy greater than V_w and they are therefore likely to be emitted. The sharp rise in thermionic emission with temperature is therefore to be attributed to a correspondingly sharp rise with temperature of the number of electrons having properly directed energies greater than V_w. Values of A and b for commonly used thermionic emitters are listed in Table 15.01.[9]

TABLE 15.01.—CONSTANTS OF THERMIONIC EMISSION

Thermionic emitter	Tempera-ture, °abs.	A, amp. per m² per degree²	b °abs.	V_w volts
Carbon.....................	2000	6.02×10^5	4.65×10^4	00
Molybdenum...............	2000	6.02×10^5	5.17×10^4	44
Platinum	1600	6.02×10^5	5.91×10^4	08
Barium oxide on platinum....	1200	2.88×10^6	1.96×10^4	68
Thorium oxide on platinum..	2000	5.7×10^3	3.70×10^4	18
Tungsten.................	2000	6.02×10^5	5.26×10^4	52
Thoriated tungsten	1600	3×10^5	3.06×10^4	2.63

Although the energy of the thermionic electrons within the metal must be sufficient to overcome the work function V_w, the energy of the electrons after emission is relatively low. Thus they tend, in the absence of other electrodes, to congregate in a cloud or *space charge* near the surface of the thermionic cathode. Since the space charge produces a field tending to repel the subsequently emitted electrons, there is a steady-state distribution of the space charge. If this is disturbed by the presence of a second electrode (anode), maintained at a positive potential

with respect to the cathode, electrons are attracted to the anode. Thus there is a steady-state limiting condition for two electrodes; the rate of emission of electrons from the cathode is precisely equal to the rate of absorption of electrons by the anode. The negative of this rate of flow is called the *anode-cathode current* or more commonly the *plate current*.

The steady-state magnitude of the anode–cathode current in a diode is related to the magnitude of the electric field at the surface of the cathode, *if the supply of electrons is practically unlimited.* * This can be demonstrated as follows.[10] Assume as a first approximation that all thermionic electrons are ejected in such a manner that their velocities, attributable to thermionic energy, are zero just outside the surface. If the potential gradient (*i.e.*, the electric field) at the surface of the cathode were positive *toward the cathode*, more electrons would be taken from the cathode, *i.e.*, the current would increase. If the field were positive away from the cathode, there would be no current, because the energy of the thermionic electrons would not be sufficient to overcome such a barrier. It follows that a steady current flows from anode to cathode only if the gradient at the cathode surface is (approximately) zero. The consequences of this phenomenon for a particular case are discussed briefly below.

If the anode and cathode of a diode are parallel planes, the space-current density is perpendicular to their surfaces, except possibly near the edges of the planes. Edge effects are here neglected. It follows that the space-charge density ρ is a function of only one spacial variable x, the distance from the cathode along the x axis, which is perpendicular to the cathode and anode. A potential diagram for the space between the cathode C and the anode A is shown in Fig. 15.01, *left;* the connections of the electrodes to external batteries are shown in Fig. 15.01, *right.* For low velocities ($v < < c$)

$$(15.03) \qquad eV = \tfrac{1}{2}mv^2 \qquad \text{or} \qquad v = \quad_{m} \qquad \text{meters per second}$$

in which V is the potential for any x; note the assumption that $V = 0$ at the cathode surface. The potential V and the charge

* Most commonly used diodes, except x-ray tubes, have this *space-charge limited* current.

density ρ are related in the region between the electrodes by Poisson's equation (Sec. 10.07)

(15.04) ϵ_0

and the current density $\mathbf{J}$ is

(15.05) $\mathbf{J} = \rho\mathbf{u}$

amperes per square meter in the direction of increasing x; $\mathbf{u}$ being the drift velocity of the electrons in the interelectrode
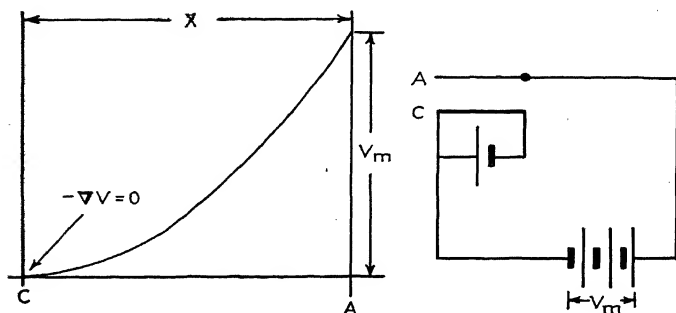
FIG. 15.01.

space. Since the electric field $\mathbf{E}$, the current density $\mathbf{J}$, and the electron velocity $\mathbf{u}$ are all parallel to the x axis, the vector notation can in this case be dropped and Poisson's equation reduces to

(15.06) $\nabla^2 V = \frac{d^2 V}{dx^2} = -\frac{\rho}{\epsilon_0}$

Then Eqs. (15.03), (15.06), and (15.05) can be combined to obtain

(15.07) $\frac{d^2 V}{dx^2} = \frac{1}{\epsilon_0}\left(\frac{m}{2e}\right)^{\frac{1}{2}} J(V)^{-\frac{1}{2}}$

Now multiply both sides by dV/dx and write the equivalent relation

(15.08) $\frac{1}{2}\frac{d}{dx}\left(\frac{dV}{dx}\right)^2 = \frac{1}{\epsilon_0}\left(\frac{m}{2e}\right)^{\frac{1}{2}} J\left[2\frac{d}{dx}(V)^{\frac{1}{2}}\right]$

because J does not vary with x in the steady-state condition; if J did vary with x, charge would accumulate or be lost at some point between the electrodes. Equation (15.08) can be integrated directly

$$(15.09) \qquad \frac{1}{2}\left(\frac{dV}{dx}\right)^2 = \frac{1}{\epsilon_0}\left(\frac{m}{2e}\right)^{\frac{1}{2}} J\,(2V^{\frac{1}{2}}) + C_1$$

Since dV/dx ($\nabla V = -\mathbf{E}$) is zero for $x = 0$, at which point $V = 0$ also, $C_1 = 0$. Thus taking the square root of Eq. (15.09) and integrating from $x = 0$ to $x = X$ ($V = 0$ to $V = V_M$)

$$(15.10) \qquad \frac{dV}{dx} = \left(\frac{8m}{\epsilon_0^2 e}\right)^{\frac{1}{4}} J^{\frac{1}{2}} V^{\frac{1}{4}}$$

$$(15.11) \qquad V^{-\frac{1}{4}}\,dV = \left(\frac{8m}{\epsilon_0^2 e}\right)^{\frac{1}{4}} J^{\frac{1}{2}}\,dx$$

$$(15.12) \qquad \frac{4}{3}\,V_M^{\frac{3}{4}} = \left(\frac{8m}{\epsilon_0^2 e}\right)^{\frac{1}{4}} J^{\frac{1}{2}} X$$

Therefore the space-charge-controlled current density J is

$$(15.13) \qquad J = \frac{4}{9}\sqrt{\frac{2e}{m}}\,\frac{\epsilon_0 V_M^{\frac{3}{2}}}{X^2} \qquad \text{amperes per square meter}$$

Thus under the ideal conditions here assumed, and approximately for most practical cases, the space-charge-limited current in a diode varies as the $\frac{3}{2}$ power of the difference of potential between the anode and cathode and inversely as the square of the distance of separation of the two elements.[*]

15.03. Note on Amplifiers, Oscillators, Detectors, and Modulators.—It is the purpose of this section to discuss qualitatively some of the properties of vacuum tubes having more than two electrodes. The quantitative analysis of such devices is the subject of courses on electronics and communication circuits.

In a three-electrode tube, or *triode*, a grid is placed between the cathode and anode (plate). In a triode *class A_1 amplifier circuit*, the grid is maintained at a potential that is negative with respect to the cathode by means of a battery or other d-c source; it thus produces a field opposed to the motion of electrons coming from the cathode toward the plate. If a source of alternating voltage,

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[*] Compare with the characteristic of the diode discussed in Chap. IX.

whose peak value is less than the magnitude of the d-c grid bias, is connected to the grid-cathode circuit, *the plate current changes as the grid voltage varies.* If there is a *load resistor R* in series with the plate circuit, such variation in the plate current i_p produces a varying output voltage Ri_p between the terminals of the load resistor. This output voltage can be as much as ten times as large as the input (grid) voltage; and the output voltage can be directly proportional, at each instant, to the input voltage. The source which provides the grid voltage usually supplies a negligible amount of power because the grid is always negative with respect to the cathode. Thus the class A_1 triode amplifier is a device which converts a low, varying voltage into a much higher varying voltage of the same wave form; practically no power is taken from the source, and the amplification is accomplished by means of energy converted from the batteries in the amplifier circuit.

The triode can be used as an *oscillator, i.e.,* a generator of alternating voltages or currents. For such service, a tuned circuit, resonant to the frequency to be generated, can be connected in series with the plate circuit. If a switch is closed in the plate circuit, a transient oscillatory current of the form given by Eq. (4.55) flows in the tuned circuit. By means of coupling—inductive or capacitative—a fraction of the varying voltage in the tuned circuit can be coupled to the grid; this causes an amplified version of this voltage to be applied to the tuned circuit. Thus, by this principle of *feedback*, the oscillation is maintained. Alternating-current energy can be taken directly from the terminals of the tuned circuit or from a circuit coupled to it.

Some of the properties of the diode are illustrated in Sec. 15.02 and in Chap. IX. The nonlinear property of the diode, illustrated by the example in Fig. 9.02, is the basis for the use of certain diodes as detectors of radio signals. Such a signal can be considered as the combination of three voltages having angular frequencies near ω_c, which is called the *carrier:*

1. *Lower side band* of angular frequency $(\omega_c - \omega_s)$.
2. *Carrier* of angular frequency ω_c.
3. *Upper side band* of angular frequency $(\omega_c + \omega_s)$.

The carrier frequency $\omega_c/2\pi$ is a few hundred kilocycles for the usual broadcasting stations in the United States. The signal

frequencies for such stations are $30 \leq \omega_s/2\pi \leq 5000$ cycles per second. When a voltage, made up of the carrier and two side bands, is applied to a diode and a resistor connected in series, *one component of the current in the circuit has a frequency $\omega_s/2\pi$,* and the other components can be effectively suppressed. By means of coupling circuits and amplifiers (see above), the audio-frequency component $(\omega_s/2\pi)$ can be applied to a loud-speaker, which transforms it into sound. Thus a detector transforms a radio-frequency combination of signals* into an audio-frequency current, which, in turn, can be transformed into sound.

The process of modulation, which is inverse to detection, consists in supplying sinusoidal voltages of angular frequencies ω_c and ω_s to a device that transforms them into sinusoidal voltages of angular frequencies $(\omega_c - \omega_s)$ and $(\omega_c + \omega_s)$, together with the carrier ω_c. Such *modulators* or *converters* are usually multi-electrode tubes; they are used in radio transmitters and also in some receivers.†

In certain kinds of service triodes are unsatisfactory because (1) the capacitance between grid and plate is large enough to produce unwanted feedback from plate to grid or (2) changes in plate voltage caused by changes in plate current affect adversely the electric field and therefore the magnitude of the plate current In such cases, the addition of one or more grids between the control grid and the plate, and the connection of these grids into the circuit, often enhance and extend the properties of the triode described above. Examples are the four-electrode and five-electrode tubes known as *tetrodes* and *pentodes*.

15.04. The Photoelectric Effect.—It is the purpose of this section to discuss the simpler aspects of the experimental observations of the ejection of electrons from metallic surfaces by light. This *photoelectric effect* is apparently inexplicable in terms of the

* All three of the radio-frequency components listed above are sent out by the usual radio broadcasting transmitter, and received by the conventional receiver. It is possible, however, to transmit only (1) *or* (3), to supply (2) at the receiver, and to detect the audio-frequency signal; this procedure is called *single-side-band* transmission.

† The discussion here applies specifically to a kind of modulation known as *amplitude modulation*, because the amplitude of the transmitted radio signal varies with the amplitude of the audio-frequency signal. A system in which the frequency or phase of the carrier is caused to vary with the amplitude of the audio-frequency signal is said to be *frequency-* or *phase-modulated*.

electromagnetic theory of radiation; the quantum theory was invoked by Einstein to account for the experimental data.

As noted in the last section, free electrons in a metal must acquire eV_w joules, directed normal to the surface of the metal, if they are to escape through the surface. Thus, if light causes photoelectrons to be emitted, the work function V_w must be overcome, according to this theory, by energy acquired by the electron from the light falling on the surface of the metal. As noted above, the work function depends critically upon the state and past history of the surface. Therefore it is important first to study the photoelectric effect in its simplest aspects, *i.e.*, the effects for clean metal surfaces in a vacuum.
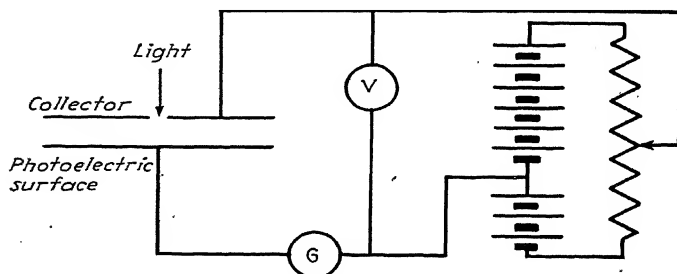
FIG. 15.02.

The maximum energy of photoelectrons ejected from a metallic surface in a vacuum can be measured by applying a difference of potential between this surface and a second parallel plate within the vacuum tube, the latter being maintained at a negative potential with respect to the former as shown in Fig. 15.02. If a beam of monochromatic light falls upon the photoelectric surface and causes the emission of photoelectrons, a current flows around the circuit unless the negative potential of the second plate is sufficiently high to prevent the arrival of photoelectrons at that plate. It is found that there is a critical voltage V_0 (< 0) below which no photoelectric current is detectable *for a particular monochromatic beam of light.* Furthermore, if current flows, its magnitude increases with V until finally the current reaches a maximum value, beyond which a further increase in V produces no appreciable increase in photoelectric current. This saturated current, presumably including all photoelectrons ejected

the surface, is directly proportional to the intensity of the monochromatic beam of light.

These experiments have important implications. The maximum-energy electrons are ejected by light of a particular frequency regardless of the intensity. According to the wave theory of light, the power per unit area, *i.e.*, the intensity, of the beam at the photoelectric surface is uniformly distributed over the surface. At very low intensities this would lead to the inference that an electron would have to store energy for a long period of time in order to acquire enough to be ejected; this not only appears logically unlikely, but it is contrary to experimental results. Thus some discontinuous theory of light is apparently required. Einstein pointed out that Planck's quantum theory of heat could be used to account for the observed phenomena of photoelectricity. Thus if monochromatic light consists of *quanta* or *photons* of energy, all having energy proportional to the frequency of the light as measured spectroscopically, then the energy of a photon might be transferred "instantaneously"* to an electron. According to this theory, if it requires V_{max} volts between the plates of the apparatus shown in Fig. 15.03 to prevent all photoelectrons from reaching the collector when the photoelectric surface is illuminated by light of frequency ν, then

$$(15.14) \qquad\qquad h\nu = \qquad\qquad eV'_w \qquad\qquad \text{joules}$$

in which h is Planck's constant whose measured value is 6.55×10^{-34} joule-seconds, ν is the frequency of the monochromatic light related to the velocity of light c and the wave length λ by $\nu = c/\lambda$ cycles per second, e is the charge of the electron which is ejected by the quantum $h\nu$, and V'_w is *approximately* equal to the work function of the photoelectric surface.† Thus, even if the beam is of low intensity from the macroscopic point of view, so many quanta will be involved that there is a high probability of the ejection of a measureable number of electrons of energy eV_{max} in accord with experimental results. For example, suppose that the beam from a 100-watt incandescent lamp is split in a spectroscope with its associated diaphragm

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* *I.e.*, in a very short unknown time interval.

† If the surfaces of the collector and emitter are alike $V'_w = V_w$; if they are unlike V'_w is the algebraic sum of V_w for the emitter and the contact difference of potential between the two surfaces.[12]

so that 10^{-9} watt per square centimeter of green light of wave length $\lambda_g = 5,540$ angstrom units (5540×10^{-10} meter) falls upon 1 square centimeter of the emitting surface. The number of quanta per second falling on this surface is

$$\frac{10^{-9} \dfrac{\text{joule}}{\text{sec. cm.}^2} \, 1 \text{ cm.}^2}{6.55 \times 10^{-34} \dfrac{\text{joule-sec.}}{\text{quantum}} \, \dfrac{3 \times 10^8 \dfrac{\text{m.}}{\text{sec.}}}{5540 \times 10^{-10} \text{ m.}}} \approx 3 \times 10^9 \frac{\text{quanta}}{\text{sec.}}$$

Note that for each photoelectric surface there is a *threshold* frequency ν_{min} below which no photoelectrons are emitted

(15.15) $$\nu_{min} = \frac{e}{h} V_w \qquad \text{cycles per second}$$

Note also that the energy of the term eV_{max} of Eq. (15.14) is equal to the kinetic energy $\frac{1}{2}mv_{max}^2$ of the photoelectrons emitted at the maximum velocity v_{max}.

This completes the introductory notes on the photoelectric effect. *Photoelectric cells*, operating on the principle of that shown in Fig. 15.02, are commonly used; they are called *vacuum photocells*. The emitting surface is often a thin deposit of cesium on oxidized silver. The sensitivity of such surfaces is a critical function of both the degree of purity of the materials of which they are made and of the manufacturing process by which they are produced. An inert gas such as argon at low pressure (of the order of 1 millimeter of mercury) is sometimes inserted in photoelectric cells to amplify the photoelectric current that would be produced in a vacuum cell. The photoelectrons, accelerated by the field of the anode-cathode difference of potential, interact with neutral gas atoms. For certain easily obtainable combinations of gas pressure and field strength such atoms may be *ionized, i.e.,* they may be split into an electron and a positive ion, which is the remainder of the atom. Thus the electron produced by such an ionizing collision becomes indistinguishable from a photoelectron. Gas amplification of 7 to 10 is commonly attained. These *gas photocells* and the various *barrier cells* and photoconductive devices are described in one of the references.[13]

15.05. The Restricted Theory of Relativity.—When a body travels at a velocity greater than about 10 per cent of the velocity

of light, the Newtonian laws of dynamics are not in accord with experimental data. Calculations according to the *theory of relativity* are in accord with these data. Furthermore, the Newtonian laws turn out to be approximations derivable from the theory of relativity for the special cases involving low velocities.

Electrons in x-ray tubes and in the betatron, and protons and ions in the cyclotron sometimes travel with velocities nearly as great as the velocity of light. Therefore the trajectories and transit times of electrons and ions in these devices must be calculated from the theory of relativity. This section consists of a brief introduction to this theory. Its chief objective is to develop those concepts which are applied in subsequent sections of this chapter to the analysis of the devices mentioned above.

The mass of an electron at rest relative to the "observer" is

$$(15.16) \qquad\qquad m_0 = 9.035 \times \qquad\qquad \text{kilogram}$$

This *rest mass* can be calculated by measuring the charge-to-mass ratio e/m by noting the deflections of electrons moving relatively slowly (< 0.1 the velocity of light) in electric or electromagnetic fields[1] and by measuring the electronic charge e by the Millikan oil-drop experiment.[2] When electrons traverse a region in a vacuum tube under the action of a difference of potential of 1,000,000 volts, the final velocity might be expected to be

(15.17)

$$\left(\frac{2eV}{m}\right)^{\frac{1}{2}} \stackrel{?}{=} \left(\frac{2 \times \qquad \times \qquad {}^{6}}{9.035 \times}\right)^{\frac{1}{2}} \stackrel{?}{=} 5.93 \times \qquad \text{meters per second}$$

i.e., about twice the velocity of light. Since there is no experimental evidence of a measured velocity greater than the measured velocity of light in a vacuum ($c = 2.998 \times 10^8$ meters per second), the method of calculation of Eq. (15.17) apparently requires critical analysis. The basis of such a critical analysis—Einstein's restricted theory of relativity—is discussed briefly in this section; the relativistic solution of the problem described above is presented in Sec. 15.06.

In its simplest form, the theory of relativity is concerned with two coordinate systems moving uniformly with respect to each other. Assume that the right-hand cartesian coordinate system $x'y'z'$ is moving with respect to a similar system xyz with a velocity v, as indicated in Fig. 15.03. Such coordinate systems are postulated in order to specify the *relations* of the physical measurements of such quantities* as length, time,

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* Which are defined by the operations that lead to their measurements in terms of numbers and agreed-upon units.

and velocity; these relations are the mathematical symbols of physical events.   Now there is nothing to substantiate the idea that there is one "correct" or absolute frame of reference that should be used to describe a group of phenomena.   Yet it appears that an observer in the $x'y'z'$ system would measure a velocity c' meters per second for a beam of light parallel to $O'X'$ whose source is fixed with respect to him while an observer in the xyz system would measure a velocity $(c + v)$ meters per second for the same beam.   These results can be calculated as follows.   In the $x'y'z'$ system
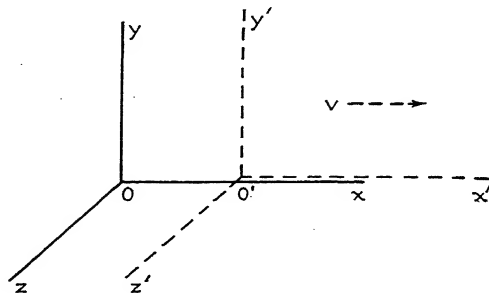
FIG. 15.03.

the velocity measured by two events—the arrival of the beam at x_2' at time t_2' and its arrival at x_1' at time t_1'—is

(15.18) $$c' = \frac{x_2' - x_1'}{t_2' - t_1'}$$ meters per second

In the xyz system it would appear that

(15.19) $\begin{aligned} x &= x' + vt' \\ y &= y' \\ z &= z' \\ t &= t' \end{aligned}$ (if $x = x'$ at $t' = 0$)

so that the velocity of the light beam in the xyz system would be

(15.20) $c = \dfrac{x_2 - x_1}{t_2 - t_1} = \dfrac{x_2' + vt_2' - x_1' - vt_1'}{t_2' - t_1'} = c' + v$ meters per second

Einstein assumed that these appearances are wrong.   He set out to find a set of transformations analogous to Eq. (15.19) upon the following assumptions, which are in accord with known experiments:

1. Measurements of the velocity of light in vacuum are independent of the relative uniform motion of the observer and the source of light.

2. Mathematical relations representing physical phenomena, devised by two observers in systems with relative uniform motion, have the same form.

Corollary assumptions of the restricted theory of relativity here discussed are:

1. Two of the required relations to replace Eq. (15.19) are $y = y'$ and $z = z'$.

2. Space and time are homogeneous and space is isotropic so that the other two transformations are of the linear form

$$(15.21) \qquad x' = ax + bt \qquad \text{and} \qquad t' = \alpha t + \beta x$$

This is unadulterated (though extremely intelligent) guessing. If the logical consequences of these guesses are borne out by experiment, the theory is good. A few of the consequences are discussed below.

The propagation of a flash of light at the instant when the coordinate systems are coincident would be specified by the two observers by the equations

$$(15.22) \qquad x^2 + y^2 + z^2 = c^2t^2$$

and

$$(15.23) \qquad x'^2 + y'^2 + z'^2 = c^2t'^2$$

so that

$$(15.24) \qquad x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$$

Substituting from Eq. (15.21) in the right-hand member of Eq. (15.24)

$$(15.25) \quad x^2 + y^2 + z^2 - c^2t^2 = a^2x^2 + 2abxt + b^2t^2 + y^2 + z^2$$
$$- a^2c^2t^2 - 2c^2\alpha\beta xt - c^2\beta^2x^2$$

Since this holds for all x, y, z, t, the coefficients of like terms are equal:

$$(15.26) \qquad \begin{aligned} a^2 - c^2\beta^2 &= 1 \\ ab - c^2\alpha\beta &= 0 \\ b^2 - a^2c^2 &= -c^2 \end{aligned}$$

Next note that there is a time t_1 for which the zero of the primed system is coincident with x_1 of the unprimed system. Then

$$x_1' = 0 = ax_1 + bt_1 \qquad \text{[from Eq. (15.21)]}$$

so that

$$(15.27) \qquad \frac{x_1}{t_1} = v = -\frac{b}{a}$$

The solutions of Eqs. (15.26) and (15.27) are

$$(15.28) \qquad \begin{aligned} \alpha &= \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} & \beta &= -\frac{v}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \\ a &= \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} & b &= -v\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \end{aligned}$$

so that the transformations are

$$x' = k(x - vt)$$

(15.29) $$y' = y \qquad\qquad k \qquad \left(1 - \frac{v^2}{c^2}\right)^{\cdot}$$

$$t' = k\left(t - \frac{v}{c^2}x\right)$$

If these are solved for x, y, z, t in terms of x', y', z', t' the results are

$$x = k(x' + vt')$$
$$y = y'$$
(15.30) $$z =$$

Thus, noting that v of Eq. (15.29) and v of Eq. (15.30) are negatives of each other, it is concluded that x, y, z, t have the same mathematical form in the two systems, *i.e.*, they are *invariant* under this transformation.

If the observer O in the unprimed system can specify two points x_2, x_1, at time t, he will assume that the distance separating them is

$$(x_2 - \qquad : S.$$

If the observer O' in the primed system can specify the *same two points* as x_2', x_1', he will calculate from Eq. (15.30) that they are separated by a distance

$$x_2' - x_1' = (x_2 - x_1)/k = S/k.$$

i.e., by a distance shorter than that calculated by the observer O. On the other hand, if O' measures a distance $x_4' - x_3' = S$, the observer O will calculate that the length is $(x_4' - x_3')/k = S/k$. Thus their is an apparent contraction of length in the direction of relative motion. Similarly there is an apparent contraction of time intervals calculated in the related systems.

The transformations of velocity from one system to the other are easily calculable. For example, suppose O measures a velocity $u_x = dx/dt$. From Eq. (15.30)

(15.31) $$dx = k(dx' + v\,dt')$$
$$dt = k\left(dt' + \frac{v}{c^2}dx'\right)$$

so that

(15.32) $$\frac{dx}{dt} = \frac{dx' + v\,dt'}{dt' + \frac{v}{c^2}dx'} = \frac{u_x' + v}{1 + \frac{u_x'v}{c^2}}$$

The observer O', measuring the velocity of light in his system in the direction x', obtains $u_x' = c$, according to the hypothesis upon which these calculations are based. Thus even for the limiting case where O' is moving along x

with a velocity $c(v = c)$, the observer O measures the velocity

(15.32-a) $$u'_x + v \qquad c + c$$ meters per second

Thus c is a limiting velocity according to the restricted theory of relativity. *No velocity measurement in any system can exceed c according to this theory.*

If the velocity u has components along the y and z axes, the additional relations similar to Eq. (15.32) above are

(15.33) $$dy \qquad\qquad dy'$$
$$\left(dt' + \frac{v}{c^2}\, dx' \right)$$
$$dz \qquad\qquad dz' \qquad\qquad u_z$$

The corresponding primed u's in terms of the unprimed u's are

(15.34) $$u_x - v$$
$$1 - \frac{u_x v}{2}$$

$$k \left(1 - \frac{u_x v}{\hat{}} \right.$$

Components of acceleration can be calculated in a similar manner.[3]   These aspects of the theory need not be traced further for present purposes.

These drastic changes of the concepts of space and time might lead to the suspicion that mass varies with velocity.   One possible guess is that, since no velocity has been measured greater than the velocity of light in a vacuum, the inertial mass m of a body varies with the velocity v of the body in such a manner that $m \to \infty$ as $v \to c$.   Thus, retaining the classical concept of force F

(15.35) $$F = \frac{d}{dt}\,(mv)$$ newtons

the deduction from this guess is that an infinite force would be required to give a body a velocity c.   The inertial masses of electrons traveling at high velocities (of the order of $0.90\,c$ to $0.98\,c$), calculated from experimental data,[4] are found to be in accord with this guess.   The experimental evidence suggests that if the mass of a body is m_0 when it is at rest relative to the

observer, its mass m when it travels with a velocity v measured in the same system is

$$(15.36)^* \qquad m = \frac{m_0}{\left(1 - \dfrac{v^2}{c^2}\right)^{\frac{1}{2}}} = km_0 \qquad \text{kilograms}$$

The derivation of this relation from the analytical point of view is given in a reference.[5]

The kinetic energy W of a body which starts from rest and acquires a velocity v is

$$(15.37) \quad W = \int_0^x F \, dx = \int_0^x \frac{d}{dt}(mv) \, dx = \int_0^x m \frac{dv}{dt} \, dx + \int_0^x v \frac{dm}{dt} \, dx$$

and since $dx = v \, dt$

$$(15.38) \qquad W = m_0 \int_0^v \frac{v \, dv}{\left(1 - \dfrac{v^2}{c^2}\right)^{\frac{1}{2}}} + \frac{m_0}{c^2} \int_0^v \frac{v^3 \, dv}{\left(1 - \dfrac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

If the second term on the right-hand side of Eq. (15.23) is integrated by parts

$$(15.39) \ \frac{m_0}{c^2} \int_0^v \frac{v^3 \, dv}{\left(1 - \dfrac{v^2}{c^2}\right)^{\frac{3}{2}}} = \frac{m_0 v^2}{\left(1 - \dfrac{v^2}{c^2}\right)^{\frac{1}{2}}} \Bigg]_0^v - 2m_0 \int_0^v \frac{v \, dv}{\left(1 - \dfrac{v^2}{c^2}\right)^{\frac{1}{2}}}$$

the remaining integrals can be combined and integrated directly. The final result is

$$(15.40) \qquad W = m_0 c^2 \left[\frac{1}{\left(1 - \dfrac{v^2}{c^2}\right)^{\frac{1}{2}}} - 1 \right] = c^2(m - m_0) \qquad \text{joules}$$

Now if this logical result of the guess represented by Eq. (15.36) is to be of practical value, W should reduce to $\frac{1}{2} m_0 v^2$ for $v \ll c$. The term $1/k$ can be expanded in a series in powers of v^2/c^2 so that another form of Eq. (15.40) is

$$(15.41) \quad W = -m_0 c^2 + \frac{m_0 c^2}{\left(1 - \dfrac{v^2}{c^2}\right)^{\frac{1}{2}}} = -m_0 c^2 + m_0 c^2 \left(1 + \frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} \right.$$

$$\left. + \frac{5}{16} \frac{v^6}{c^6} + \cdots \right)$$

$$W = \frac{1}{2} m_0 v^2 + \frac{3}{8} m_0 \frac{v^4}{c^2} + \frac{5}{16} m_0 \frac{v^6}{c^4} + \cdots$$

which reduces to the classical expression for kinetic energy if $v \ll c$.

* See Problem 15.01, page 412.

If there is a definite energy $W = c^2(m - m_0)$ corresponding to a *change* in mass, $(m - m_0)$, it might be inferred that mass and energy are different aspects of the same thing* and, furthermore, that the total energy W' represented by a mass m is

(15.42)   joules

The effects of a transformation of a quantum of radiant energy into two subatomic entities—a positron and an electron—and the effects of the reverse transformation have been observed.[6] The calculations from these observations are in accord with Eq. (15.42). These phenomena, however, are not relevant to the subjects discussed in this chapter.

15.06. Introduction to Electron Ballistics; X-ray Tube.—It is the purpose of the next five sections to discuss briefly several problems related to the motion of electrons and ions in five commonly used devices: the x-ray tube, two forms of the cathode-ray oscilloscope, the cyclotron, and the betatron.

Before describing the motions of electrons in x-ray tubes, a few properties of electromagnetic radiation and of x-ray tubes are discussed briefly in the following paragraphs.

X-rays are radiant energy produced when high-velocity electrons are suddenly decelerated. The kinetic energy of *the stream* of moving electrons is apparently transformed during deceleration chiefly into heat, only a small portion of the kinetic energy (of the order of 0.01 per cent) being transformed into x-rays; the kinetic energy of *a few* of these electrons may, however, be entirely transformed into radiant energy in the form of x-rays. The x-ray tube is one of the least efficient of all electric devices.

X-rays have many properties commonly attributed to heat, light, ultraviolet radiation, and the γ-rays emitted by radioactive elements. Thus they occupy a part of the so-called *electromagnetic spectrum*. Just as the wave length of light can be calculated from the results of experiments made with a prism or diffraction grating, so the wave lengths of x-rays can be calculated from the results of experiments made with crystals used as diffraction gratings. The data of Table 15.02 are intended to show representative ranges of wave length and frequency for

* *I.e.*, that mass plus energy is conserved, though mass or energy is not. Incidentally, it is the authors' opinion that if the postulates of, and the derivations from, the theory of relativity could have been written in terms of new quantities instead of in terms of classical names such as mass, time, length, velocity, etc., the theory would be more quickly intelligible to the student.

TABLE 15.02.—EXAMPLES OF ELECTROMAGNETIC RADIATION*

(Velocity in a vacuum, $c = 2.998 \times 10^8$ m. per sec.)

Name	Sources	Uses	Commonly used wave lengths, m.	Frequencies corresponding to these wave lengths, sec.$^{-1}$	Energy of one quantum, joule
"Radio broadcasting" signals.	Vacuum-tube oscillators and antennas	Radio transmission of speech and music	600 to 200	5×10^5 to 15×10^5	3.3×10^{-28} to 9.9×10^{-28}
"Short-wave" radio signals.	Vacuum-tube oscillators and antennas	Radio transmission of code signals, speech, and music (also medical diathermy[14])	40 to 10	75×10^5 to 3×10^7	49.1×10^{-28} to 2.0×10^{-26}
"Television".	Vacuum-tube oscillators and antennas	Radio transmission of pictures, speech, and music	6 to 3	5×10^7 to 10^8	3.3×10^{-26} to 6.6×10^{-26}
"Ultra-high frequencies" or "micro-waves."	Specially designed vacuum tubes and radiators[18]	Special signaling procedures (war secrets)	3 – ?	10^8 – ?	6.6×10^{-26} – ?
Heat.	Combustion, electric heaters, etc.	Metallurgy, cooking, etc.	2×10^{-6} to 8×10^{-7}	1.5×10^{14} to 3.75×10^{14}	9.8×10^{-20} to 24.5×10^{-20}
Light (visible radiation)	Electric arcs, incandescent lamps, etc.	Illumination	8×10^{-7} to 4×10^{-7}	3.75×10^{14} to 7.5×10^{14}	24.5×10^{-20} to 49×10^{-20}
Ultraviolet radiation	Mercury arcs, glow discharge tubes	Therapy of lesions of the skin, fluorescent tubes, etc.	4×10^{-7} to 10^{-7}	7.5×10^{14} to 3×10^{15}	49×10^{-20} to 196×10^{-20}
X-rays	High-voltage vacuum diode	Therapy of deep-seated lesions, medical and industrial radiography	10^{-10} to 10^{-12}	3×10^{18} to 3×10^{20}	19.6×10^{-16} to 19.6×10^{-14}
γ-rays	Naturally and artificially radioactive elements	Therapy of superficial lesions	10^{-11} to 5×10^{-13}	3×10^{19} to 6×10^{20}	19.6×10^{-15} to 39.3×10^{-14}

* Note that only representative commonly used ranges of wave length (and frequency) are listed in this table, and also that the very important uses of radiation in physical research are omitted from the table.

each of several parts of the electromagnetic spectrum. The energy of one quantum, corresponding to each frequency in the table, is listed in the last column. As noted in the last section this energy W is

$$(15.43) \qquad\qquad W = h\nu \qquad\qquad \text{joules per quantum}$$

A small number of electrons in an x-ray tube is so decelerated that all their energy is transformed into x-rays as noted above. Since the energy acquired by an electron of charge e falling through a difference of potential V is eV joules, the quantum $h\nu_{max} = hc/\lambda_{min}$ of x-rays produced by one such event is specified by

$$(15.44) \qquad\qquad h\nu_{max} = \frac{hc}{\lambda} = eV \qquad\qquad \text{joules}$$

so that the minimum wave length λ_{min} is

$$(15.45) \qquad\qquad \lambda_{min} = \frac{}{V} \qquad\qquad \text{meters}$$

The student should calculate from Eq. (15.45) several peak x-ray tube voltages V required to produce the x-rays having several of the minimum wave lengths from 10^{-10} meter to 10^{-12} meter as listed in Table 15.02.

The x-ray tubes commonly used today are diodes. The thermionic cathode is usually a tungsten filament operating at 4 to 10 volts, 3 to 6 amperes. The anode is often a truncated cylinder of copper about 4 centimeters in diameter, having embedded in the truncated end a disk of tungsten 1 to 2 millimeters thick; the angle between the surface of the tungsten disk and a plane perpendicular to the axis of the cylindrical glass tube in which the anode and cathode are mounted is from 15 to 45 degrees. A cross section of an x-ray tube is shown in Fig. 15.04.

Electrons emitted from the tungsten filament (cathode) are accelerated toward the anode. Experiment shows that the following assumptions are satisfied at least to the degree necessary for attaining satisfactory accuracy from the practical point of view: (1) The thermionic electrons are emitted with zero velocity from the incandescent filament; and (2) all electrons that finally reach the anode acquire energy eV joules, where V is the potential difference between anode and cathode. Note, however, that if

a high-frequency alternating voltage is applied to the tube, the transit time of the electrons may be an appreciable part of the half-period of the source, so that the assumption 2 may then be invalid. An example is given below to illustrate this point. Since most x-ray tubes are supplied with direct current or recti-
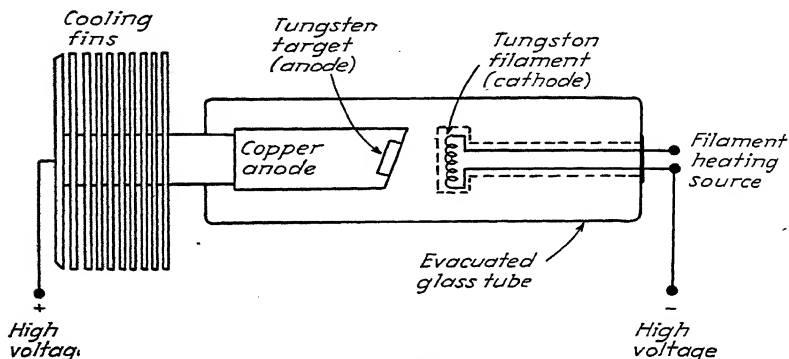
FIG. 15.04.
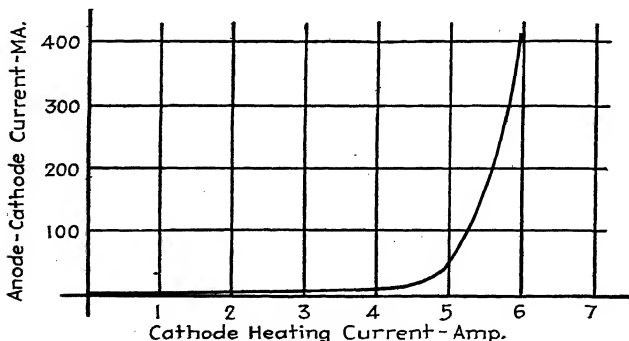
FIG. 15.05.

fied 60-cycle alternating current, this point need not generally be considered.

If the field strengths between anode and cathode are so high that thermionic energies can be neglected (assumption 1 above), it follows that the current through the x-ray tube consists of practically all the thermionic electrons that are emitted by the

filament. The magnitude of the anode-cathode current is therefore controlled by adjusting the temperature of the cathode. The relation between the cathode current and the average x-ray tube current for a typical tube supplied with full-wave rectified 60-cycle alternating current is shown in Fig. 15.05.

If the x-ray tube supply is rectified alternating current, the current reaches saturation only during a small fraction of each half cycle. Figure 15.06 shows the oscillograms of the anode-cathode voltage and the x-ray tube current for a tube supplied with full-wave rectified 60-cycle alternating current.

Since x-ray tube voltages are commonly from 10 to 1,000 kilovolts-peak or more, the electrons in them commonly travel with velocities of the same order of magnitude as the velocity of light. In order to calculate their velocities when they arrive at the anode, the relativistic formulas of Sec. 15.04 must be used. Thus if the electron traverses a difference of potential V, it acquires energy eV, which is equal, according to Eq. (15.40), to

$$(15.46) \qquad eV = m_0 c^2 \left[\frac{1}{\quad} - 1 \right] \qquad \text{joules}$$

in which v_T is the velocity when the electron strikes the anode. Solving for v_T

$$(15.47) \qquad v_T = \left[\frac{}{eV} - 2m_0 c^2 \right]^{\frac{1}{2}} \qquad \text{meters per second}$$

If the terminal velocity v_T is very small with respect to c, *i.e.*, if $eV <<< m_0 c^2$, Eq. (15.47) reduces to

$$(15.48) \qquad v_T = \qquad \text{meters per second}$$

i.e., to the result obtained by equating $eV =$ ·

The equation of motion of an electron in an x-ray tube can be simply derived if it be assumed that the electron is subjected to a constant force eE throughout its traverse of the distance $d = V/E$ from cathode to anode. Thus

$$(15.49) \qquad d$$

TIME – SECONDS
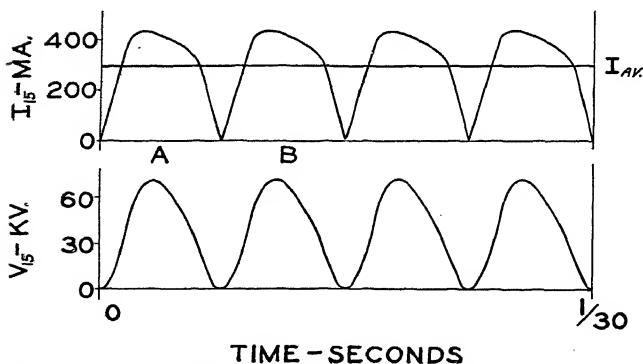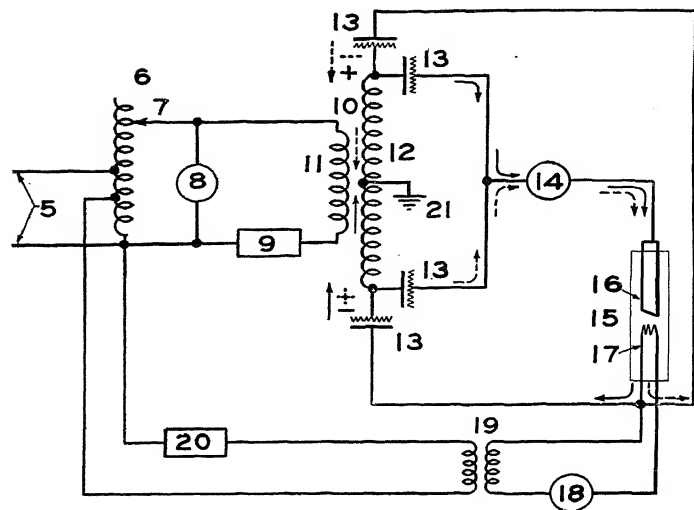
FIG. 15.06.—Full-wave rectified x-ray (diagnostic) apparatus using four thermionic rectifying tubes: 5, source of 220-volt, 60-cycle a-c; 6, auto-transformer; 8, pre-reading voltmeter; 9, exposure timer; 10, high-tension transformer; 20, cathode–current regulator; 19, cathode heating transformer; 14, ballistic milliampere–second meter; 15, x-ray tube of which 16 is the anode and 17 is the cathode; 18, cathode-current ammeter; 21, ground; 13, thermionic rectifiers; V_{15}, oscillogram of x-ray tube voltage for two cycles; I_{15}, oscillogram of x-ray tube current for two cycles. Average x-ray tube current $I_{AV} = 300$ ma. (*From "Radiologic Physics"; C. Weyl, S. R. Warren, Jr., D. B. O'Neill* (1941); *courtesy of the publisher, Charles C. Thomas, Springfield, Ill., and Baltimore, Md.*)

so that

$$(15.50) \qquad\qquad mv = eEt$$

the constant of integration of Eq. (15.50) being zero because it is assumed that $v = 0$ at $t = 0$. Putting in the value of m from Eq. (15.21), and writing $v = dx/dt$, the following relation is obtained

$$(15.51) \qquad \frac{dx}{dt} = \frac{ecEt}{(m_0^2c^2 + e^2E^2t^2)^{\frac{1}{2}}} \qquad \text{meters per second}$$

Integration of Eq. (15.51) leads to a relation between x and t; this involves a constant of integration, which is the site of the electron at $t = 0$; the value is $x = 0$ at $t = 0$ so that

$$(15.52) \qquad\qquad\qquad\qquad eE \qquad\qquad\qquad \text{meters}$$

The particular value of t, say T, which is calculated from Eq. (15.52) for $x = d$ and $E = V/d$ is the *transit time* for an electron traveling from cathode to anode.

$$(15.53) \qquad\qquad T = \qquad\qquad\qquad\qquad\qquad \text{second}$$

The remainder of this section deals with an example of an x-ray tube operating at 1,000 kilovolts-peak. Assume that a series of tubular accelerating anodes along a cylindrical tube 1.22 meters long are so connected to the high-voltage source that the force on the electrons is nearly constant along this length $d = 1.22$ meters,* and that the electrons "coast" at constant velocity v_T from the last accelerating anode to the target.[15] The arrangement is indicated in Fig. 15.07. The terminal velocity v_T is 2.83×10^8 meters per second or $0.94c$ from Eq. (15.47). The transit time T from a to b (Fig. 15.07) is 0.58×10^{-8} second from Eq. (15.53). From b to c the electron travels with constant velocity 2.83×10^8 meters per second along a path 0.81 meter long, so that its transit time T' from b to c is 0.29×10^{-8} second. Thus the total transit time from cathode to target is 0.87×10^{-8} second. The maximum-energy

* Note in reference 15 that the potential gradient is *not* constant along the length of the tube so that the calculation given in this section is an approximation.

quanta and the minimum wave length of the x-rays produced by such electrons are 1.59×10^{-13} joule and 0.012×10^{-10} meter from Eqs. (15.44) and (15.45).

Problem: Calculate the ratio of the force eE for the 1,000-kvp. tube discussed above to the gravitational force on an electron.
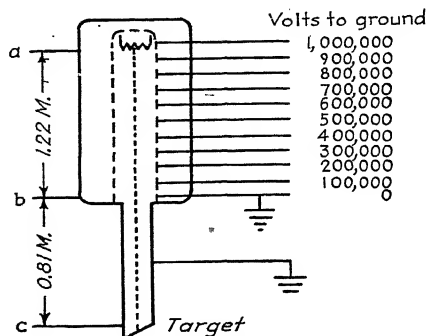
FIG. 15.07.
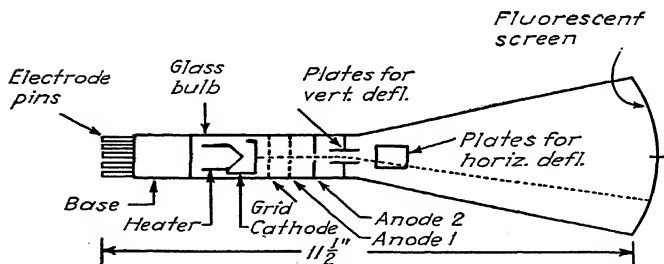
FIG. 15.08.

15.07. Electron Ballistics; Cathode-ray Tube with Electrostatic Deflecting Plates.—The *oscilloscope* discussed in this section consists of an *electron gun*, two pairs of electrostatic deflecting plates, and a fluorescent screen, arranged as shown in Fig. 15.08 in cross section. The electron gun consists of the *heater* and associated thermionic cathode, a grid, and two anodes. Its purpose is to eject through anode 2 a beam of electrons coincident with the axis of the tube. The number of electrons per unit length of the beam is controlled by adjusting the potential of the grid with respect to the cathode. The speed of the elec-

trons as they leave the gun is determined chiefly by the potential of anode 2 with respect to the cathode. The intermediate electrode, anode 1, is so arranged geometrically that its potential with respect to anode 2 controls the focusing of the beam to a relatively small area (of the order of 0.5 millimeter in diameter) on the fluorescent screen at the end of the tube. The fluorescent material (usually calcium tungstate or zinc sulphide) is placed on the inside of the end of the tube; the spot of fluorescent light produced where the electron beam impinges on the *phosphor* (the fluorescent material) is visible from the outside of the tube.
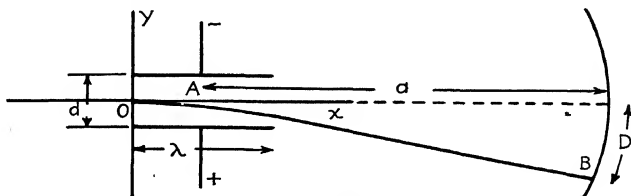
Fig. 15.09.

A detailed drawing of the path of the electron beam between one pair of deflecting plates is shown in Fig. 15.09. Two assumptions are made to simplify the calculation of the deflection of the beam by a voltage V applied to the deflecting plates: (1) The field between the deflecting plates is uniform and equal to V/d volts per meter, *i.e.*, edge effects are neglected; (2) the deflection $D = a \tan \theta$, *i.e.*, the curvature of the fluorescent screen is small. Note that for $x > \lambda$ the electrons travel in a straight line with a constant velocity $(v_x^2 + v_y^2)^{\frac{1}{2}}$ at an angle θ with respect to the axis of the tube.

The equations of motion need not be put in the relativistic form for most commonly used tubes because the electron velocities are very small compared to the velocity of light. If V_1 is the accelerating voltage applied to anode 2 and cathode, the x component of velocity is, from Eq. (15.48)

$$(15.48a) \qquad\qquad v_x = \qquad\qquad \text{meters per second}$$

everywhere to the right of $x = 0$. Thus, if $t = 0$ is taken as the time when an electron enters the space between the plates

$(x = 0)$, the subsequent x coordinate of the electron is

$$(15.54) \qquad\qquad x = \sqrt{\frac{2eV_1}{m_0}}\, t \qquad\qquad \text{meters}$$

It follows that the electron is subjected to a force V/d from $t_0 = 0$ to $t_1 = \sqrt{\frac{m_0}{2eV_1}}\,\lambda.$ During this interval

$$(15.55) \qquad\qquad d \; {}_{(m,n)} = \frac{eV}{} \qquad\qquad \text{newtons}$$

so that

$$(15.56) \qquad\qquad \frac{dy}{} = \frac{eV}{}\, t \qquad\qquad \text{meters per second}$$

for $v_y = 0$ at $t = 0$ and

$$(15.57) \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{meters}$$

since $y = 0$ at $t = 0$.   Eliminating t from Eqs. (15.54) and (15.57) gives the parabolic path of the electron while it is between the plates:

$$(15.58) \qquad\qquad y = \qquad\qquad \text{meters}$$

and the direction of motion dy/dx for $x = \lambda$ is

$$(15.59)$$

Note that

$$\text{meters}$$

Thus the equation of the line AB is

$$(15.61) \qquad\qquad y = \quad^{V\lambda}\quad x - \qquad\qquad \text{meters}$$

This line intersects the x axis at $x = \lambda/2$ and it intersects the screen at B for which

$$(15.62) \qquad\qquad = D = V\lambda a \qquad\qquad \text{meters}$$

Therefore the deflection is proportional to the applied voltage V and the sensitivity is

$$(15.63) \qquad\qquad \frac{D}{V} \qquad\qquad \text{meters per volt}$$

As a practical example suppose that $a = 20$ centimeters, $\lambda = 1.6$ centimeters, $d = 0.5$ centimeters, $V_1 = 1,000$ volts. The sensitivity calculated from Eq. (15.63) is 0.32 millimeter per volt. If $V_1 = 500$ volts, the sensitivity is 0.64 millimeter per volt.

The second pair of plates produces deflections at right angles to that produced by the first set. If the pair that produces vertical deflection of the beam be connected to a source of periodic voltage having a period T second, and if the pair which produces horizontal deflections be connected to a source that alone would cause the spot to go across the screen from left to right with uniform velocity in a time interval of about $0.9T$ and then back to the left in $0.1T$, the periodic voltage would be represented on the screen in the same form as a graph of the voltage.[16]

15.08. Electron Ballistics; Cathode-ray Tube with Electromagnetic Deflecting Coils.—Two current-carrying coils, one on either side of the tube, mounted so that their axis is perpendicular to the paper at A, Fig. 15.09, will cause the beam to be deflected upward or downward as indicated in that figure. Using the symbols shown in Fig. 15.09, and assuming that the magnetic field through the paper is uniform over the rectangle λ, d and proportional to the current in the coils, the approximate deflection is calculated as follows. The linear velocity of an electron remains constant while it traverses the magnetic field, because the effect of the field is to exert a force on the electron perpendicular to the direction of its motion. Thus, from the instant the electron leaves the electron gun until it strikes the screen, its linear velocity is

$$(15.64) \qquad\qquad \sqrt{\frac{2eV_1}{m_0}} \qquad\qquad \text{meters per second}$$

While it is traversing the region in which there is a uniform

magnetic field B webers per square meter, it is subjected to a force $\mathbf{F}$

(15.65) $$\mathbf{F} = e(\mathbf{v} \times \mathbf{B})$$ newtons

Since this force is always perpendicular to $\mathbf{v}$ and since $\mathbf{B}$ is assumed to be constant in magnitude throughout λ, d, the electron must move in a circle and the force must be directed toward the center of the circle; this center clearly lies on the line YO extended downward, Fig. 15.09. If the electron travels in a circular arc, its centrifugal force m_0v^2/r is equal in magnitude to the magnitude of $\mathbf{F}$, so that

(15.66) $$evB = \frac{m_0v^2}{r}$$

The radius of the circular path is therefore

(15.67) $$r = \frac{m_0v}{Be}$$ meters

or

(15.68) $$r = \frac{1}{B}\left(\frac{2m_0V_1}{e}\right)^{\frac{1}{2}}$$ meters

The equation of the circle is

(15.69) $$x^2 + y^2 = \frac{2}{B}\left(\frac{2m_0V_1}{e}\right)^{\frac{1}{2}} y$$

The intersection of this circle with the line $x = \lambda$ determines where the electron leaves the magnetic field. The deflection y_λ for this condition is the smaller solution of

(15.70) $$y_\lambda^2 - \frac{2}{B}\left(\frac{2m_0V_1}{e}\right)^{\frac{1}{2}} y_\lambda + \lambda^2 = 0$$

which is

(15.71) $$y_\lambda = \frac{1}{B}\left(\frac{2m_0V_1}{e}\right)^{\frac{1}{2}} - \left(\frac{2m_0V_1}{B^2e} - \lambda^2\right)^{\frac{1}{2}}$$ meters

The slope of the path of the electron as it leaves the field B is, from Eqs. (15.69) and (15.71)

(15.72) $$\frac{dy}{dx}\bigg]_{x=\lambda} = \tan\theta = \frac{\lambda}{\left(\dfrac{2m_0V_1}{B^2e} - \lambda^2\right)^{\frac{1}{2}}}$$

and the deflection D is

$$(15.73) \quad D = \frac{\left(a - \dfrac{\lambda}{2}\right)\lambda}{\left(\dfrac{2m_0V_1}{B^2e} - \lambda^2\right)^{\frac{1}{2}}} + \frac{1}{B}\left(\frac{2m_0V_1}{e}\right)^{\frac{1}{2}}$$

$$- \left(\frac{2m_0V_1}{b^2e} - \lambda^2\right)^{\frac{1}{2}} \quad \text{meters}$$

In most practical cases $\lambda << 2a$ and $\lambda^2 << 2m_0V_1/B^2e$ so that a good approximation for D is

$$(15.74) \qquad D \approx a\lambda B\left(\frac{e}{2m_0V_1}\right)^{\frac{1}{2}} \qquad \text{meters}$$

It is clear from the material of Chap. XIII that it is impossible to produce a uniform magnetic field over a rectangle d, λ which is zero elsewhere in the plane of the rectangle. Thus the formulas calculated above are useful chiefly for calculating the order of magnitude of the deflection D. For example, if a pair of coils carrying a current I produces an approximately uniform field $B = 10^{-3}I$ webers per square meter through a rectangle of length $\lambda = 1.6$ centimeters, in a cathode-ray tube for which $a = 20$ centimeters and $V_1 = 1,000$ volts, the deflection per unit current would be of the order of 0.03 millimeter per milliampere.

15.09. The Cyclotron.—The *cyclotron*[17] is a device for accelerating positive ions to high velocities. It has been used for experiments in the field of nuclear physics, *i.e.*, for disintegrating atomic nuclei. The details of its construction and use are extremely complicated. The purpose here is to describe briefly the principles of its operation.

The cyclotron consists of an evacuated space in which there is a source of ions and two D-shaped electrodes; a constant magnetic field in the space is produced by an external electromagnet. The apparatus is shown diagrammatically in Fig. 15.10. The electromagnet NS produces a field B in the direction shown in Fig. 15.10, *left* (B is normally outward from the paper in Fig. 15.10, *right*). The source of ions is at O. High-frequency alternating voltage from a vacuum-tube oscillator is connected to the dees at 1, 2. If there is an electric field between the dees when the ion is between them, the ion is accelerated toward

one of them.   The electric field *inside* the dees is zero, so that the ion describes a semicircular path when it enters one of the dees (see last section).   When it emerges from this dee it is accelerated toward the other, if the polarity of the alternating voltage supplied to the dees has reversed.   The process is repeated again and again until the ion emerges at T having acquired energy equal to the product of its charge times the effective accelerating voltage applied to it during one traverse the dees *times the number of times it crosses* the effective voltage between dees is 10^5 volts ana the ion crosses the space 100 times, the final velocity of the ion is the same as that which it would have attained by falling through a potential difference of 10 million volts.
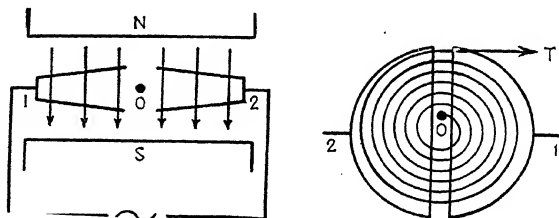
Fig. 15.10.

This device will work only if the time required for an ion to traverse a semicircular path within a dee is independent of its velocity when it enters, so-that its arrivals at the space between the dees are separated by precisely one half period of the a-c voltage applied to the dees.   As noted in Sec. 15.07 if a charge q of mass m and velocity v traverses a constant magnetic field B, the force mv^2/r is equal to the electromagnetic reaction Bqv (if B and v are perpendicular to each other as in this case), and q travels in a circle.   Now if the charge describes a circle of radius r with a velocity v, the time T required for one revolution is

(15.75) $$T = \frac{2\pi r}{\ddot{}}$$ seconds per revolution

or the angular velocity ω is .

(15.76)   radians per second

But from Eq. (15.67) of the last section, r for this case can be expressed as a function of m, B, v, q.   Substituting in Eq. (15.76)

$$(15.77) \qquad \frac{}{m} \qquad \text{radians per second}$$

Therefore, although the linear velocity v of the ion in the cyclotron is increased every time it traverses the space between the dees, and although the radius of the semicircle it traverses subsequently is larger than the semicircle it traversed before, its angular velocity while it is in the dees is constant because ω is dependent only upon B, q, m.*

Suppose that the diameter of the dees is 1.5 meters, the effective voltage between the dees 60 kilovolts, and that hydrogen ions ($q = 1.59 \times 10^{-19}$ coulomb, $m = 1.66 \times 10^{-27}$ kilogram) are released 5 centimeters from the center of the dees.   Assume further that the space between the dees is small enough compared to the semicircles described by the ions so that the time taken to go from one dee to the other can be neglected.   The flux density B is assumed to be 0.7 weber per square meter.   The high-frequency supply to the dees must have a frequency f

$$\frac{}{} = 10.7 \quad \text{megacycles}$$

The radius of the first semicircle would be

$$\frac{1}{B} \qquad = 5.05 \qquad \text{centimeters}$$

Since each trip between the dees adds velocity corresponding to V_1, the successive radii $r_2, r_3, r_4, r_5, \ldots$ will be

$$r_n = 5.05 \sqrt{n} \qquad \text{centimeters } (n = 2, 3, 4, \qquad k)$$

with centers within a few centimeters of the center of the cyclotron.   Thus k, the number of trips between the dees, will be approximately the value of n for which r_n is equal to the radius

* But note that the acceleration of the least massive of positive ions—the proton—by equivalent voltages of the order of 50 million volts or more would be so great that relativistic calculations would be required.   The constancy of angular velocity derived above holds only for nonrelativistic circumstances.

the direction of the arrow E$_-$, Fig. 15.11.   Its magnitude is

$$(15.79) \qquad e. \qquad \frac{e}{2\pi r} \frac{\partial \phi}{\partial t} \qquad \text{newtons}$$

e the momentum of the electron at any instant by p;
acting on the electron is equal to the rate of change of
momentum

$$(15.80) \qquad eE = \frac{e}{2\pi r} \frac{\partial \phi}{\partial t} = \frac{\partial p}{\partial t} \qquad \text{newtons}$$

that the electron starts
there is no flux through t
$= 0$ at $t = 0$); the integral of E   then

$$(15.81) \qquad\qquad\qquad\qquad\qquad\qquad \text{newton-seconds}$$

If the electrons injected into the tube at $t = 0$ are to traverse a
of radius R_0, the centripetal and centrifugal forces
The radial force outward
neous angular velocity ω
equal to

orbit

(15.82)

or

$$(15.83) \qquad\qquad \frac{\mu}{B_0} \doteq$$

i.e., the instantaneous field at the orbit must be proportional to
the instantaneous momentum if the electrons are to continue on a
stable orbit of radius R_0.   The condition for stability is obtained
by equating values of p from Eqs. (15.81) and (15.83)

$$(15.84) \qquad\qquad \phi = 2\pi R_0^2 B_0 \qquad\qquad \text{webers}$$

The electromagnet must be designed so that Eq. (15.84) is
satisfied *for all instants of time* during the acceleration of the
pulse of electrons.

* Practically, the electrons are injected into the tube with a speed corre-
sponding to a few kilovolts during a short (about 8-microsecond) time
interval when the flux ϕ is slightly more than zero.

In the 20-million-volt betatron the magnet winding is supplied with 180-cycle a. c.   The electrons are injected for an interval of 8 microseconds just after the flux has passed through zero and is increasing.   Acceleration continues throughout nearly one-quarter cycle ($\frac{1}{720}$ second); the field is suddenly decreased as the flux reaches its peak value and the electron orbit expands, causing the electrons to strike the target.

From Eq. (15.47) the velocity of an electron corresponding to 20 million volts is only one-tenth of 1 per cent less than the velocity of light.   During the period of acceleration—about $\frac{1}{720}$ second—electrons travel about 400,000 meters, making nearly 1 million revolutions about the stable orbit, whose radius is about 18 centimeters.

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Problems

15.01. The mass m of an object that is at rest in a system of coordinates that travels at velocity v with respect to the observer is

$$m = \frac{m_0}{\left(1 - \dfrac{v^2}{c^2}\right)^{\frac{1}{2}}}$$

where m_0 is the rest mass. Show that, if the object is acted upon by a force F_t at right angles to $\mathbf{v}$, the force is

$$F_t = m_t \frac{dv}{dt}$$

in which $m_t = m$. Show that, if the object is subjected to a force F_l in the direction of $\mathbf{v}$, the force is

$$F_l = m_l \frac{dv}{dt}$$

in which

$$m_l = \frac{m_0}{\left(1 - \dfrac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

The quantity m_t is often called the *transverse mass;* the quantity m_l is the *longitudinal mass.*

15.02. If a positron and an electron are converted into two equal quanta, what are the energies of these quanta?

15.03. If a positron–electron pair—each traveling with a velocity of 3×10^7 m. per sec.—is formed by the conversion of a maximum-energy quantum from an x-ray tube, what is the peak voltage applied to the x-ray tube?

15.04. What are the maximum thermionic currents that can be obtained from one side of a flat tungsten cathode 3 by 10 mm. at temperatures of 1000° abs., 1500° abs., 2000° abs., 2500° absolute?

15.05. A diode consists of the cathode described in Problem 15.04 and an anode of the same area parallel to the cathode, at a distance of 2 mm. If the cathode is maintained at a uniform temperature of 2000° abs., what currents flow with anode voltages of 25 volts, 50 volts, 75 volts, 100 volts, 200 volts?

15.06. Calculate the minimum wave length of x-rays produced by anode-cathode peak voltages of 8 kvp., 40 kvp., 100 kvp., 200 kvp., 600 kvp., 1,000 kvp., 5,000 kvp.; and calculate the velocity of the electrons as they arrive at the anode in each case.

15.07. An x-ray tube operates with a d-c voltage of 200 kv. and a d-c current of 18 ma. Calculate approximately the rate at which transformer oil should be circulated through the interior of the anode to prevent its temperature from exceeding 100°C., with an ambient temperature of 25°C.

15.08. A photoelectric cell has a threshold wave length of 6,300 Å. (1 Å $= 10^{-10}$ m.). Calculate the maximum velocity of photoelectrons that are emitted when the incident light has a wave length of 4,000 Å.

15.09. The work function of lithium is 2.35 volts. Neglecting the contact difference of potential between anode and cathode, calculate the maximum velocity of photoelectrons ejected by a lithium-cathode photocell illuminated by monochromatic light of wave length 3959 Å.

15.10. A nine-stage *electron multiplier* (RCA type 931) consists of a photoemissive surface and nine *dynodes*. Light falling on the photoemissive surface causes the emission of photoelectrons; these are accelerated toward the first dynode; when they strike this dynode, a larger number of *secondary electrons* are emitted and accelerated toward the second dynode; the process is repeated, until finally all electrons are collected by the eleventh electrode, the anode. At normal operating conditions (125 volts per stage) the ratio of the anode current to the photoelectric current at the cathode is 230,000. What is the average number of secondary electrons produced at a dynode by one primary electron impinging on that dynode?

15.11. The x-ray tube, whose cathode characteristics are shown in Fig. 15.05, is operated from an a-c source with a nominal rms voltage of 230 volts, using an x-ray tube current of 300 ma. If variations of x-ray tube current of ± 10 ma. are practically permissible, what variations of input voltage can be permitted? If the input-voltage variations are greater, what procedure do you recommend for maintaining the variations of x-ray tube current within the limits of ± 10 ma.?

15.12. Electrons are projected by an electron gun G into the space between two parallel plates maintained at potentials 0 and V volts as indicated

in the figure below.   The plane of the electron trajectory is perpendicular to the plates.   Calculate the trajectories for the following conditions, and plot these trajectories on cross-section paper:

   a. $V = -100$ volts; $\theta = 45$ deg.; $v_0 = 6 \times 10^5, 6 \times 10^6, 6 \times 10^7$ m. per sec.

   b. $V = 100$ volts; $\theta = 45$ deg.; initial velocities as in *a.*

   c. $V = -100$ volts; $v_0 = 6 \times 10^6$ m. per sec.; $\theta = 0, 30, 60, 90$ deg.
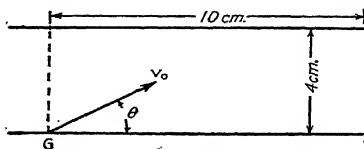
PROB. 15.12.

15.13. In a particular cathode-ray tube (RCA type 905) one pair of electrostatic deflecting plates have their centers 8 in. from the fluorescent screen; their capacitance is 2 $\mu\mu$f.   The sensitivity is 0.19 mm. per volt, for electrons having velocities corresponding to 2,000 volts.   Calculate approximately the dimensions of the rectangular deflecting plates and their spacing.

15.14. What is the trajectory of an electron beam that is projected with a velocity corresponding to V volts into a uniform magnetic field of flux density B webers per sq. m. at an angle of θ deg. with respect to the magnetic field?

CHAPTER XVI

MAXWELL'S EQUATIONS
AND ELECTROMAGNETIC RADIATION

16.01. Maxwell's Equations.[1]—From the relations developed in the preceding chapters the following differential equations may be formulated for points in free space at which no charges exist, and they are found to lead to results consistent with experimentally observed phenomena. In these equations the symbols have the following significance:

$\mathbf{E}$ = electric-field intensity (vector sum of the electrostatic and electromotional intensity) at a given point P at a given time t,

$\mathbf{H}$ = magnetic-field intensity (magnetizing force) at this point—at this time,

ϵ_0 = permittivity of free space (8.85×10^{-12} farad per meter)

μ_0 = permeability of free space (1.257×10^{-6} henry per meter).

The basic differential equations are then

(16.01) $$\nabla \cdot \mathbf{E} = 0$$

(16.02) $$\nabla \cdot \mathbf{H} = 0$$

(16.03) $$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

(16.04) $$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

The first of these equations arises from the fact that from the microscopic point of view $\mathbf{D} = \epsilon_0 \mathbf{E}$ and $\nabla \cdot \mathbf{D}$ is always equal to the charge density ρ at this point. Therefore if ρ is zero, $\nabla \cdot \mathbf{D} = \epsilon_0 \nabla \cdot \mathbf{E}$ is likewise zero. The second of these equations arises from the fact that from the microscopic point of view, irrespective of the presence or absence of a charge density, $\mathbf{B} = \mu_0 \mathbf{H}$ and $\nabla \cdot \mathbf{B} = \mu_0 \nabla \cdot \mathbf{H}$ is zero. The third of these equations arises from the fact that in a homogeneous medium $\mathbf{B}$ is always equal to $\mu \mathbf{H}$, and $\nabla \times \mathbf{E}$ is equal $-\partial \mathbf{B}/\partial t$ [see Sec. 12.04,

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Eq. (12.19)].   Hence in free space

$$\nabla \times \mathbf{E} = - \frac{\partial}{\partial t}(\mu_0 \mathbf{H}) = -\mu_0 \frac{\partial \mathbf{H}}{\partial d}$$

The fourth equation is a special case of the more general relation that

$$(16.05) \qquad \oint_C \mathbf{H} \cdot d\mathbf{s} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{\partial}{\partial t} \int_S \mathbf{D} \cdot d\mathbf{S}$$

Neglecting the second integral in the right-hand member, the reader will recognize this relation as the well-known principle that, when varying electric fields are neglected, the line integral of the magnetizing force **H** around a closed loop C is equal to all the conduction current that threads any surface bounded by this
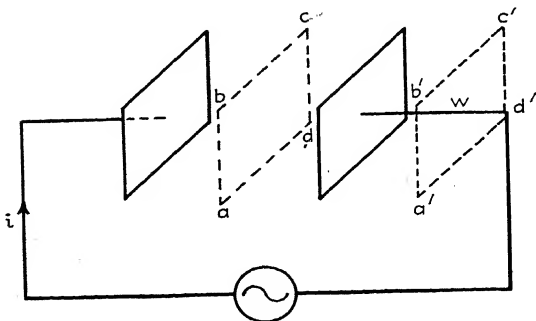
Fig. 16.01.

loop (see Sec. 12.06). The addition of the second integral in the right-hand member was suggested by Maxwell, in order to introduce a term to take care of the fact that a closed loop C may be the boundary of a surface through which there is no conduction current but around which the line integral of **H** is *not* zero; see, for example, the loop *abcd* in Fig. 16.01, which is a diagrammatic sketch of a capacitor connected to a source of varying emf. The plane surface indicated by the dotted lines is not threaded by any conduction current, but a surface bounded by the dotted lines $a'b'c'd'$ and pierced by one of the wires at W, Fig 16.01, *is* threaded by a conduction current.[2] The addition of this second integral

makes the *total* current, the density of which is

$$(16.06) \qquad\qquad \frac{\partial}{\partial t} \qquad \text{amperes per square meter}$$

continuous in the sense that its value is the same for *every* surface bounded by a loop such as *abcd* or *a'b'c'd'*.

The quantity

$$\frac{\partial}{\partial t} \int_S \mathbf{D} \cdot \mathbf{dS}$$

was called by Maxwell the *displacement current,* and $\partial\mathbf{D}/\partial t$ the *density* of the displacement current.   The following quotation is from Maxwell's treatise,[3] published in 1873:

610. One of the chief peculiarities of this treatise is the doctrine which it asserts, that the true electric current $\mathfrak{C}$, that on which the electromagnetic phenomena depend, is not the same thing as $\mathfrak{K}$, the current of conduction, but that the time-variation of $\mathfrak{D}$, the electric displacement, must be taken into account in estimating the total movement of electricity, so that we must write,

$$\mathfrak{C} = \mathfrak{K} + \dot{\mathfrak{D}} \qquad \text{(Equation of True Currents) (H)}$$

The four equations (16.01) to (16.04) are known as Maxwell's equations.   When proper precautions are taken in the interpretations of the symbols, a more general formulation of these equations, applicable to material media, is the following:

$$(16.07) \qquad\qquad \boldsymbol{\nabla} \cdot \mathbf{D} = \rho$$
$$(16.08) \qquad\qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0$$
$$(16.09) \qquad\qquad \boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$(16.10) \qquad\qquad \boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Maxwell's equations themselves do not include the so-called *force equation*

$$(16.11) \qquad\qquad F = q(\mathbf{u} \times \mathbf{B}) \qquad \text{newtons}$$

discussed in Sec. 14.07.   The reader should note that although $\mathbf{u} \times \mathbf{B}$ is a force per unit charge, it is not a part of the

intensity **E**, which latter is that force per unit charge exerted upon a charge dq held at *rest* at the point under consideration; attributable to the positions and motions of *other* charges. The force **u** $\times$ **B** per unit charge, which may be thought of as a *transverse electromotional intensity*, plays an important role in all problems pertaining to the mechanical forces between current-carrying conductors and the deflection of electronic and ionic beams (see Chap. XV).

16.02. Plane Waves in Free Space.—This section is devoted to the derivations of certain functions representing **E** and **H** that satisfy the Maxwell equations for *free space, i.e.,* that satisfy Eqs. (16.01) to (16.04). The special case based on the assumption that **E** and **H** are functions of x and t only is here considered.

If **E** is a function of x and t only, its three components E_x, E_y, and E_z are likewise functions of x and t only. Hence the only partial derivatives of each of these three components that have values different from zero are the partial derivatives with respect to x and t. Therefore, in this special case

$$(16.12) \qquad \boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\partial E_x}{\partial x}$$

$$(16.13) \qquad \boldsymbol{\nabla} \times \mathbf{E} = -\mathbf{j}\,\frac{\partial E_z}{\partial x} + \mathbf{k}\,\frac{\partial E_y}{\partial x}$$

Similarly

$$(16.14) \qquad \boldsymbol{\nabla} \cdot \mathbf{H} = \frac{\partial H_x}{\partial x}$$

$$(16.15) \qquad \boldsymbol{\nabla} \times \mathbf{H} = -\mathbf{j}\,\frac{\partial H_z}{\partial x} + \mathbf{k}\,\frac{\partial H_y}{\partial x}$$

Therefore, Eqs. (16.01) to (16.04) become

$$(16.16) \qquad \frac{\partial E_x}{\partial x} = 0$$

$$(16.17) \qquad \frac{\partial H_x}{\partial x} = 0$$

$$(16.18) \qquad -\mathbf{j}\,\frac{\partial E_z}{\partial x} + \mathbf{k}\,\frac{\partial E_y}{\partial x} = -\mu_0\left(\mathbf{i}\,\frac{\partial H_x}{\partial t} + \mathbf{j}\,\frac{\partial H_y}{\partial t} + \mathbf{k}\,\frac{\partial H_z}{\partial t}\right)$$

$$(16.19) \qquad -\mathbf{j}\,\frac{\partial H_z}{\partial x} + \mathbf{k}\,\frac{\partial H_y}{\partial x} = \varepsilon_0\left(\mathbf{i}\,\frac{\partial E_x}{\partial t} + \mathbf{j}\,\frac{\partial E_y}{\partial t} + \mathbf{k}\,\frac{\partial E_z}{\partial t}\right)$$

Equating like components in Eqs. (16.18) and (16.19), there

result

(16.20)
$$\frac{\partial H_x}{\partial t} = 0$$

(16.21)
$$\frac{\partial E_x}{\partial t} = 0$$

(16.22)
$$\frac{\partial E_z}{\partial x} = \mu_0 \frac{\partial H_y}{\partial t}$$

(16.23)
$$\frac{\partial E_y}{\partial x} = -\mu_0 \frac{\partial H_z}{\partial t}$$

(16.24)
$$\frac{\partial H_z}{\partial x} = -\epsilon_0 \frac{\partial E_y}{\partial t}$$

(16.25)
$$\partial x$$

Now differentiate Eq. (16.22) with respect to x and differentiate Eq. (16.25) with respect to t; equate the two values of $\partial^2 H_y / \partial x\,\partial t$ so obtained. Eliminate H_z in the same manner from Eqs. (16.23) and (16.24). The results are

(16.26)

(16.27)
$$\partial x^2 \qquad \partial t^2$$

This is a special form of the *wave equation*. Direct substitution shows that

(16.28) $E_y = A f_1(\beta x - \omega t) + B f_2(\beta x + \omega t)$ volts per meter

is a solution of Eq. (16.26) if

(16.29)
$$\frac{\omega^2}{\beta^2} =$$

Note that f_1 and f_2 are assumed to be dimensionless while A and B have the dimensions volts per meter regardless of the functional forms of f_1 and f_2. Thus, for example,

(16.30) $E_y = A \sin (\beta x - \omega t) + B \sin (\beta x + \omega t)$

is one such solution. This particular solution is discussed further in Sec. 16.04.

Consider first the function $f_1(\beta x - \omega t)$. If the argument $(\beta x - \omega t)$ of this function is $(\beta x_1 - \omega t_1)$ at a particular point

having an x coordinate x_1 at a particular instant of time t_1, then there will be a time $t_2(t_2 > t_1)$ at some point having an x coordinate $x_2(x_2 > x_1)$ for which

$$(16.31) \qquad \beta x_1 - \omega t_1 = \beta x_2 - \omega t_2$$

and E_y, so far as the function f_1 is concerned, will have the same value at x_2 at time t_2 as it had at x_1 at time t_1. Thus f_1 represents the propagation of something in the $+x$ direction with a velocity

$$(16.32) \qquad \frac{x_2 - x_1}{t_2 - t_1} \qquad 1 \qquad \text{meters per second}$$

Inserting the numerical values of μ_0 and ϵ_0, it is found that the velocity of propagation is 2.998×10^8 meters per second.

When Maxwell arrived at this result he recognized that this predicted velocity of propagation of an electromagnetic disturbance corresponded with great accuracy to the measured velocity of light. The implication that this electromagnetic theory was applicable to light and to other forms of radiation, known and unknown, was inescapable.

The function f_2 can be shown by a similar analysis to represent the propagation of a similar kind *toward* the origin with a velocity $(\mu_0\epsilon_0)^{-\frac{1}{2}}$. The commonly used symbol for this velocity is c, as noted in previous chapters. The function f_1 is called the *incident wave;* the function f_2 is called the *reflected wave.* The term *wave* represents a propagated disturbance of which Eq. (16.28) is an example; it is in this example called a *plane wave* because E and H are constant, at a particular instant, at all points in any yz plane.

The value of H_z is related to the value of E_y. Substituting Eq. (16.30) in Eq. (16.23)

$$(16.33) \qquad \beta f_1'(\beta x - \omega t) + \beta f_2'(\beta x + \omega t) = -\mu_0 \frac{\partial H_z}{\partial t}$$

If this is integrated with respect to t, the result is

$$(16.34) \qquad H_z = \frac{\beta}{\omega\mu_0} [Af_1(\beta x - \omega t) - Bf_2(\beta x + \omega t)]$$

$$H_z = \sqrt{\frac{\epsilon_0}{\mu_0}} [Af_1(\beta x - \omega t) - Bf_2(\beta x + \omega t)].$$

amperes per meter

The solution of Eq. (16.27) for E_z and the subsequent solution of Eq. (16.22) for H_y are obtained by the same methods as those described above; the student can work them out as a practice problem. The results are

(16.35) $E_z = Cf_3(\beta x - \omega t) + Df_4(\beta x + \omega t)$ volts per meter

(16.36) $H_y = \dfrac{\sqrt{}}{\mu_0}$

                                                                              amperes per meter

in which f_3 and f_4 are arbitrary functions.

16.03. The Poynting Vector.—Analysis of Eqs. (16.01) to (16.04) in the manner discussed briefly below shows that the electromagnetic disturbance discussed in Sec. 16.02 can be considered as the propagation of energy through space. First take the scalar product of Eq. (16.03) by $\mathbf{H}$ and the scalar product of Eq. (16.04) by $-\mathbf{E}$, and add:

(16.37) $\mathbf{H} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) - \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{H}) = -\mu_0 \mathbf{H} \cdot \dfrac{\partial \mathbf{H}}{\partial t} - \epsilon_0 \mathbf{E} \cdot \dfrac{\partial \mathbf{E}}{\partial t}$

By expanding the left-hand member of Eq. (16.37) it can be shown that it is identical to $\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H})$; the right-hand member is equivalent to $-\dfrac{\partial}{\partial t}\left[\dfrac{1}{2}\mu_0 H^2 + \dfrac{1}{2}\epsilon_0 E^2\right]$, so that

(16.38) $\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) = -\dfrac{\partial}{\partial t}\left[\dfrac{1}{2}\mu_0 H^2 + \dfrac{1}{2}\epsilon_0 E^2\right]$

For a volume v surrounded by a closed surface S, the integral form of Eq. (16.38) is

(16.39) $\displaystyle\int_v \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H})\,dv = \int_S (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{dS}$

$$= -\int_v \dfrac{\partial}{\partial t}\left[\dfrac{1}{2}\mu_0 H^2 + \dfrac{1}{2}\epsilon_0 E^2\right] dv$$

The term $\epsilon_0 E^2/2$ is precisely the energy per unit volume calculated in Chap. XI, Eq. (11.69), for the particular case of the space between the plates of a parallel-plate condenser. The term $\mu_0 H^2/2$ is precisely the energy per unit volume calculated from Eq. (13.81) for the case for which $B = \mu_0 H$. Therefore Eq.

(16.39) suggests the interpretation, which appears to be justified by experiment, that the surface integral of the vector $(\mathbf{E} \times \mathbf{H})$ or the (equivalent) volume integral of the scalar $\nabla \cdot (\mathbf{E} \times \mathbf{H})$ *is equal to the time rate of decrease of electromagnetic energy in the volume.* The quantity*

$$(16.40) \qquad\qquad \mathbf{P} = \mathbf{E} \times \mathbf{H} \qquad \text{watts per square meter}$$

is called *Poynting's vector* or the *power-flux density.* Note particularly that $\mathbf{P}$ may have a value in cases where there is no propagation of energy. For example,[4] if there are an electrostatic field and a magnetostatic field at right angles to each other in a region v, $\mathbf{P}$ has a value, but both the surface integral of $\mathbf{P}$ and the volume integral of $\nabla \cdot \mathbf{P}$ are zero; so also is the time rate of decrease of electromagnetic energy within the volume v.

16.04. Propagation of a Plane-polarized Plane Wave.—It can be shown[5] that a charge executing simple harmonic motion in a straight line, or a sinusoidal alternating current in a straight wire, will produce a *plane-polarized plane wave* at distances from the source that are very large with respect to the dimension of the linear source. A plane-polarized plane wave is a plane wave (see Sec. 16.02) for which, in any yz plane determined by an x measured in the direction of propagation, the vector $\mathbf{E}$ and therefore the vector $\mathbf{H}$ have the same directions *at all times.* For simplicity assume that the y axis is chosen in the direction of $\mathbf{E}$; then there is only one component H_z of the magnetic field in the region (far from the source) under consideration.

Practically, then, this simplified problem is that of an electromagnetic wave, or radio wave, produced at large distances from a straight-wire *antenna* that is connected to a source of high-frequency alternating current.

Assume that f_2 of Eq. (16.28) is zero and that $A_1 f_1$ of Eq. (16.28) is

$$(16.41) \qquad\qquad E = E_y \sin (\beta x - \omega t) \qquad \text{volts per meter}$$

It follows from the discussion of Sec. 16.02 that $E_x = E_z = 0$, that $H_x = H_y = 0$, and that

$$(16.42) \qquad\qquad H = H_z \sin (\beta x - \omega t) \qquad \text{amperes per meter}$$

---

* Do not confuse the symbol $\mathbf{P}$ here used for power-flux density with the same symbol used in Sec. 10.12 for electric polarization.

in which

(16.43)   ohms

The quantity E_y/H_z is called the *impedance of free space in the x direction*.   This concept is extremely useful in the analysis of many problems of electromagnetic radiation.[6]   No further reference to this concept is made in the analysis of this simple plane-wave problem.

Note that, just as in the case of traveling waves on transmission lines (Chap. VII), β is the *wave length constant*

(16.44)   per meter

where the wave length λ is the distance in meters, measured at a particular time t_1, from a point x_1 where the argument of the sine in Eq. (16.41) or Eq. (16.42) is $(\beta x_1 - \omega t_1)$ to a point x_2 where the argument is $(\beta x_2 - \omega t_1) = (\beta x_1 + 2\pi - \omega t_1)$.
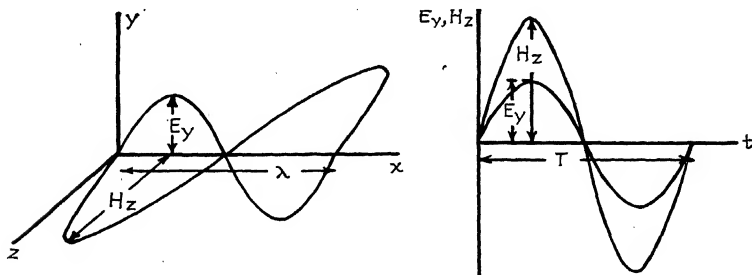
Fig. 16.02.

The constant ω is the angular frequency of the alternating current in the antenna, *i.e.*:

(16.45) $$\omega = 2\pi f = \frac{2\pi}{T}$$ per second

in which T is the period in seconds.

It follows from Eqs. (16.32), (16.44), and (16.45) that

(16.46) $c = f\lambda$ meters per second

E and H can be conveniently represented graphically in this particular problem as indicated in Fig. 16.02.

The Poynting vector **P** in this simple example is

$$(16.47) \quad \mathbf{P} = \mathbf{E} \times \mathbf{H} = iEH = iE_y^2 \sqrt{\frac{\epsilon_0}{\mu_0}} \sin^2 (\beta x - \omega t)$$

Consider now a rectangular parallelepiped having a length λ meters and a cross section of 1 square meter, as shown in Fig.
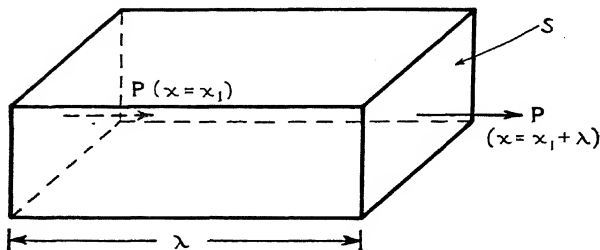
FIG. 16.03.

16.03.   The power-flux vector **P** is constant for a given x; therefore $\int_v (\boldsymbol{\nabla} \cdot \mathbf{P})\, dv$ is

$$(16.48) \quad \int_v (\boldsymbol{\nabla} \cdot \mathbf{P})\, dv = \int_1^{x_1+\lambda} \left[2\beta E_y^2 \sqrt{\frac{\epsilon_0}{\mu_0}} \sin (\beta x - \omega t) \right. $$
$$\left. \cos (\beta x - \omega t) \right] dx = \beta E_y^2 \sqrt{\frac{\epsilon_0}{\mu_0}} \int_{x_1}^{x_1+\lambda} \sin 2(\beta x - \omega t)\, dx = 0$$

i.e., the total electromagnetic energy within the volume is constant, from Eq. (16.39).   The time average of the intensity of the radiation in the x direction is

$$(16.48a) \quad \left[\int_S \right. \qquad\qquad {}_y^2 \sqrt{\frac{\epsilon_0}{\mu_0}} \sin^2 (\beta x - \omega t)\, dt$$
$$= \frac{1}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} E_y^2 \quad \text{watts per square meter}$$

Electric-field intensities of the order of 100 microvolts per meter (rms) are required for good radio reception with commonly used receiving apparatus.   This corresponds to a power-flux density of the order of 30 micromicrowatts per square meter.

Additional elementary problems in the application of Maxwell's equations are discussed in the references[7,8,9,10] listed at the end of the chapter.

16.05. Solution of Maxwell's Equation in Terms of the Vector and Scalar Potentials.—Two functions—the retarded scalar potential V and the retarded vector potential $\mathbf{A}$—are consistent with Maxwell's equations. The correlation of the Maxwell equations and the potentials is of great theoretical interest.[11] The correlation is outlined below for a region in which charge densities ρ exist in free space. The concepts of the retarded potentials are discussed further in Appendix E. The Maxwell equations are

$$(16.49) \qquad \mathbf{\nabla} \cdot \mathbf{D} = \rho$$

$$(16.51) \qquad \qquad \qquad \qquad \qquad \qquad \frac{\partial t}{}$$

$$(16.50) \qquad \mathbf{\nabla} \cdot \mathbf{B} = 0$$

$$(16.52) \quad \mathbf{\nabla} \times \mathbf{H} = \rho\mathbf{u} + \frac{\partial \mathbf{D}}{\partial t}$$

Designate by V a scalar function of the space coordinates x, y, z and of time t, and designate by $\mathbf{A}$ a vector function each of whose components A_x, A_y, A_z is likewise a function of these space coordinates and time. Impose upon these two functions the following conditions

$$(16.53) \qquad\qquad \mathbf{E} = -\mathbf{\nabla}V$$

$$(16.54) \qquad\qquad \mathbf{B} = \mathbf{\nabla} \times \mathbf{A}$$

$$(16.55) \qquad\qquad -\mu_0\epsilon_0 \frac{\partial V}{\partial t} = \mathbf{\nabla} \cdot \mathbf{A}$$

Since for any vector $\mathbf{A}$ (see Appendix B)

$$(16.56) \qquad \mathbf{\nabla} \times \mathbf{\nabla} \times \mathbf{A} = \mathbf{\nabla}(\mathbf{\nabla} \cdot \mathbf{A}) - \mathbf{\nabla}^2\mathbf{A}$$

it can be shown that the defining relations in Eqs. (16.53) to (16.55) taken in conjunction with Eqs. (16.49) to (16.52) give rise to the two differential equations (writing $c^2 = 1/\mu_0\epsilon_0$)

$$(16.57) \qquad\qquad \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} \qquad\qquad \epsilon_0$$

$$(16.58) \qquad\qquad \frac{1}{} \frac{\partial^2 \mathbf{A}}{}$$

When the *microscopic* point of view is adopted and it is assumed that no electric or magnetic field exists at an infinite distance from the origin it can be shown[11] that the solutions of Eqs.

(16.57) and (16.58) are

$$(16.59) \qquad V \qquad 1 \qquad\qquad\qquad\qquad \text{volts}$$

$$(16.60) \qquad\qquad\qquad\qquad dv \qquad \text{webers per meter}$$

The square brackets are used in Eqs. (16.59) and (16.60) to specify a particular manner of setting up the integrands. The scalar potential V is calculated for any point P at time t by taking the value of ρ in a volume dv surrounding each point P' in space *at a time* $(t - r/c)$, in which r is the distance from P to P'; the integral is evaluated by summing up ρ/r for all of space v_s. Similarly, to calculate the contribution to $\mathbf{A}$ at P at time t, the value of ρu *which was at P' at time* $(t - r/c)$ is used. The quantities V and $\mathbf{A}$ are called *retarded potentials*.

If it is possible to calculate V and $\mathbf{A}$ from Eqs. (16.59) and (16.60) —and it is always formally possible—$\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$, and $\mathbf{H}$ can be calculated from Eqs. (16.53) and (16.54), $\mathbf{H} = \mathbf{B}/\mu_0$ and $\mathbf{D} = \varepsilon_0\mathbf{E}$ (see also Appendix E).

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Problems

16.01. What is the displacement current in a condenser of capacitance 1 μf connected to a 110-volt, 60-cycle, a-c source?

16.02. Discuss the boundary conditions and associated phenomena when a plane-polarized plane wave falls normally on an infinite sheet of a perfect conductor. Discuss qualitatively the case of an imperfect conductor, such as copper or carbon.

16.03. Imagine an infinite chunk of copper with a flat surface; if the current density parallel to the surface is, at the surface, J_0 amp. per sq. m., how does J vary with depth below the flat surface?

16.04. An isolated point source of electromagnetic radiation emits energy at the rate of 1 watt. Calculate the magnitude of Poynting's vector and the maximum values of $\mathbf{E}$ and $\mathbf{H}$ at distances of 20, 40, 60, 80, 100 cm. from the source.

16.05. The four walls of a pipe[12] of rectangular cross section (a meters by b meters) are perfectly conducting. At one end of the pipe the vector $\mathbf{E}$ is parallel at all times to the side of length b; its magnitude is zero along each side of length b and maximum at the mid-point between them; the variation of E at any instant with the coordinate parallel to the sides a is a half cycle of a sine wave. The vector $\mathbf{E}$ is a sinusoidal function of time of frequency f cycles per second. Under what circumstances will electromagnetic energy be propagated down the pipe with no attenuation? What are the components of the magnetic-field intensity $\mathbf{H}$?

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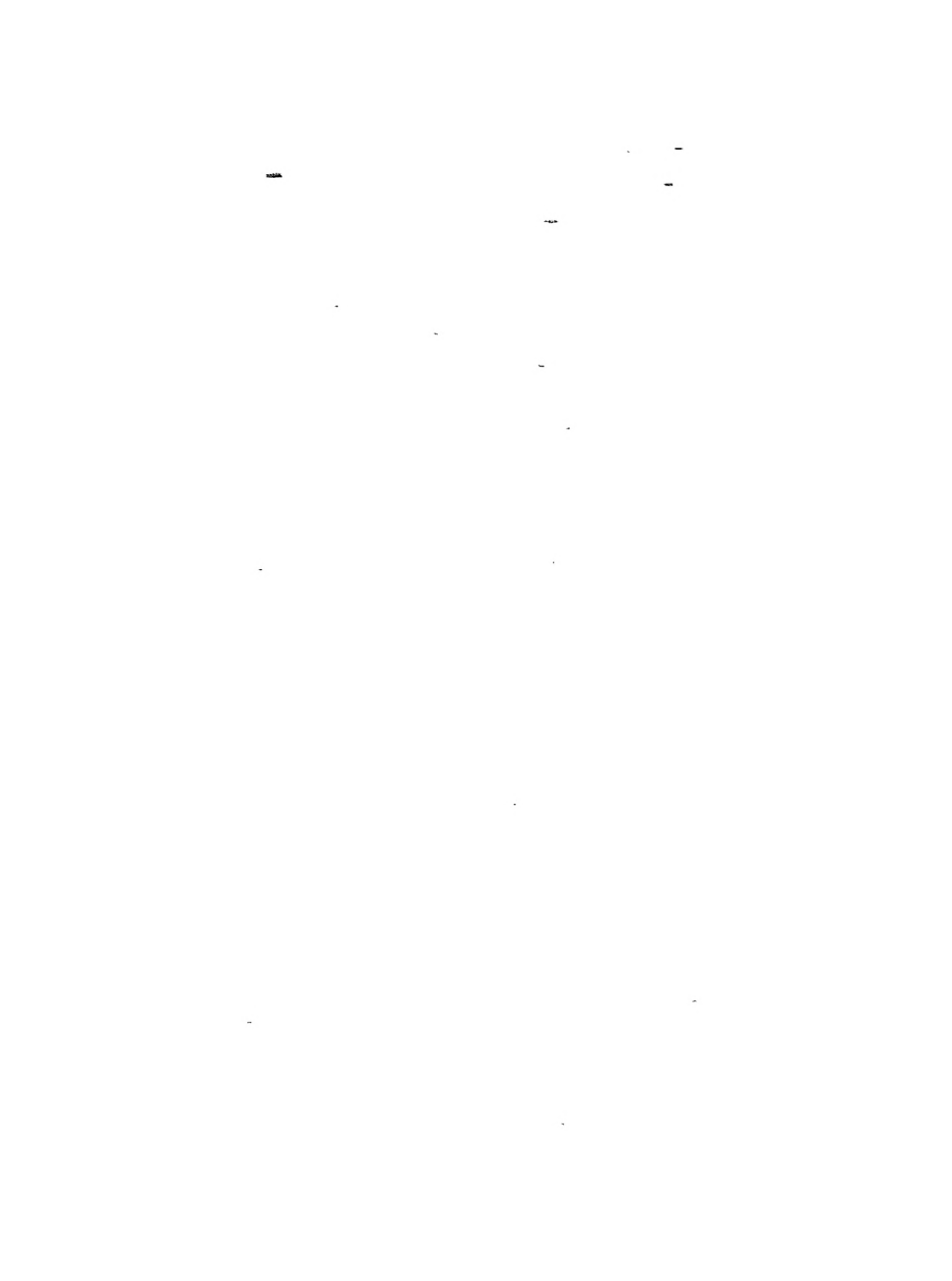
APPENDIX A

THE SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

A.01. Introduction.—In Chap. IV the solutions of several differential equations are used without benefit of derivation. This appendix consists of a brief summary of methods of solving the equations.

It should be noted at the outset that no general analytic process is defined for the process of integration, although such a process *is* defined for the process of differentiation. Thus many functions $f(x)$ have been differentiated; therefore, these functions are the indefinite integrals that are listed in tables of integrals.[1,2]

Whenever an ordinary differential equation is to be solved, the essential feature of the problem is a search for one or more functions which, when substituted in the differential equation, cause it to be reduced to an algebraic identity. Since there is no analytic procedure for integration, the solution of a differential equation may be obtained in any manner whatsoever. Thus the equation may be integrated by separating variables and using tables of known integrals, or by multiplying each term of the equation by a factor that transforms it so that tables can be used, or by substituting a new variable or variables for the old, or by assuming a solution in the form of an infinite series, or by setting up the equation on the differential analyzer,[3] . . . , or by "guessing." The importance of "guessing," based upon wide knowledge of the common integrals listed in tables and *particularly upon experience*, cannot be overemphasized.

Linear differential equations possess an important property: *if each of two or more functions satisfies the equation, the sum of these functions is also a solution.*

A.02. Equations of the First Order.—Three forms of the first-order linear differential equation with constant coefficients are derived in Chap. IV:

(A.01) $R \dfrac{dq}{dt} + \dfrac{q}{C} = 0$ $q = Q_0$ for $t = 0$

(A.02) $R \dfrac{dq}{dt} + \dfrac{q}{C} = E$ $q = Q_0$ for $t = 0$

(A.03) $R \dfrac{dq}{dt} + \dfrac{q}{C} = E \cos (\omega t + \phi_1)$ $q = Q_0$ for $t = 0$

Their solutions are derived below. Note that, throughout this appendix, the notation is that used in particular cases in Chap. IV. The student can rewrite the solution of Eq. (A.03), which is derived below, so that it applies to

(A.04) $L \dfrac{di}{dt} + Ri \doteq E \cos (\omega t + \theta)$ $i = I_0$ at $t = 0$

by substituting symbols as pointed out in Chap. IV.

The solution of Eq. (A.01) can be obtained by putting $E = 0$ into the solution of Eq. (A.02). The solution of Eq. (A.02) can be obtained by separating the variables q and t, and integrating, *i.e.*, by looking up the answers in a table of integrals. The solution proceeds as follows. Multiply through Eq. (A.02) by $C\,dt$, and group the terms in the form

(A.05) $RC\,dq = (CE - q)\,dt$

Next divide both members by the product of the coefficient (RC) of dq and the coefficient $(CE - q)$ of dt; note also that $d(CE - q) = -dq$, so that

(A.06) $\dfrac{d(CE - q)}{CE - q}$

According to the boundary conditions, this is to be integrated between limits as follows

(A.07) $\displaystyle\int_{Q_0}^{q} \dfrac{d(CE - q)}{CE - q} = -\dfrac{1}{RC} \Big] \quad dt$

and the result is

(A.08) $\log \dfrac{CE - q}{CE - Q_0} = -\dfrac{t}{RC}$

The inverse logarithms (antilogs) of the two members of Eq. (A.08) are

$$(A.09) \qquad \frac{CE - q}{CE - Q_0} = \varepsilon^{-\frac{t}{RC}}$$

so that the solution of Eq. (A.02) is

$$(A.10) \qquad q = CE(1 - \varepsilon^{-\frac{t}{RC}} \qquad \overline{RC}$$

When $Q_0 = 0$, Eq. (A.10) reduces to Eq. (4.19). When $E = 0$ and $Q_0 = CE'$, Eq. (A.10) reduces to Eq. (4.21). The derivatives of Eq. (A.10)

$$\frac{}{dt} \qquad \frac{E}{R} - \frac{Q_0}{RC} \qquad ^{-}\overline{RC}$$

reduce, with these boundary conditions, to Eqs. (4.20) and (4.22).

In order to solve Eq. (A.03) let it be assumed that the complete solution v can be expressed as the product of two functions q_1 and q_2. Assume

$$(A.12)* \qquad q = q_1 q_2$$

The result of substituting Eq. (A.12) in Eq. (A.03) is

$$q_1 \frac{dq_2}{dt} + \left(R \frac{dq_1}{dt} + \frac{q_1}{C} \right) q_2 = E \cos (\omega t + \phi_1)$$

Now further assume that

$$(A.14) \qquad R \frac{dq_1}{dt} + \frac{q_1}{C} = 0$$

so that

$$-\frac{t}{RC}$$

Inserting these results in Eq. (A.13), and solving for q_2:

$$(A.16) \qquad q_2 = \frac{E}{Q_{10}R} \int_0^t \varepsilon^{\frac{t}{RC}} \cos (\omega t + \phi_1) \, \cdot \qquad Q_{20}$$

---

* In the corresponding physical problem, q_1 or q_2 has the dimensions (coulomb)$^{\frac{1}{2}}$.

The integral in the right-hand side can be integrated by parts.

$$(A.17) \quad \int_0^t \varepsilon^{\frac{t}{RC}}[\cos (\omega t + \phi_1) \, dt] = \varepsilon^{\frac{t}{RC}}\left[\frac{\sin (\omega t + \phi_1)}{\omega} \right]_0^t$$
$$- \int_0^t \left[\frac{\sin (\omega t + \phi_1)}{\omega} \right]\left[\frac{\varepsilon^{\frac{t}{RC}}}{RC} \, dt \right]$$

The integral of the right-hand member of Eq. (A.17) can be integrated by parts also

$$(A.18) \quad \int_0^t \varepsilon^{\frac{t}{RC}}[\sin (\omega t + \phi_1) \, dt] = \varepsilon^{\frac{t}{RC}}\left[- \frac{\cos (\omega t + \phi_1)}{\omega} \right]_0^t$$
$$- \int_0^t \left[- \frac{\cos (\omega t + \phi_1)}{\omega} \right]\left[\frac{\varepsilon^{\frac{t}{RC}}}{RC} \, dt \right]$$

Combining the results of Eqs. (A.17) and (A.18), Eq. (A.16) becomes

$$(A.19) \quad q_2 = \frac{ER}{\omega Q_{10}\left(R^2 + \frac{1}{\omega^2 C^2} \right)}\left[\varepsilon^{\frac{t}{RC}} \sin (\omega t + \phi_1) \right.$$
$$\left. + \frac{1}{\omega RC} \varepsilon^{\frac{t}{RC}} \cos (\omega t + \phi_1) - \sin \phi_1 - \frac{1}{\omega RC} \cos \phi_1 \right] + Q_{20}$$

Putting this result, Eqs. (A.19) and (A.15), into Eq. (A.12)

$$(A.20) \quad q = Q_{10}Q_{20}\varepsilon^{-\frac{t}{RC}} + \frac{ER}{\omega\left(R^2 + \frac{1}{\omega^2 C^2} \right)}\left[\sin (\omega t + \phi_1) \right.$$
$$\left. + \frac{1}{\omega RC} \cos (\omega t + \phi_1) - \varepsilon^{-\frac{t}{RC}} \sin \phi_1 - \frac{\varepsilon^{-\frac{t}{RC}}}{\omega RC} \cos \phi_1 \right]$$

This is the general solution of Eq. (A.03). Note from Eq. (A.12) that $Q_{10}Q_{20} = Q_0$ is a single constant—the value of q at $t = 0$.

In Chap. IV the boundary condition for the problem involving Eq. (A.03) is $Q_{10} = 0$. This makes the first term of the right-hand member of Eq. (A.20) zero. The remaining terms can be put in the form given in Chapter IV in the following manner. First define the angle θ

$$(A.21) \qquad\qquad \theta = \tan^{-1} \omega RC$$

Rewrite Eq. (A.20), inserting functions of θ and putting

(A.22) $q = \dfrac{E}{\omega\sqrt{R^2 + \dfrac{1}{\omega^2 C^2}}}\left[\dfrac{R}{\sqrt{R^2 + \dfrac{1}{\omega^2 C^2}}}\sin(\omega t + \phi_1)\right.$

$+ \dfrac{\dfrac{1}{\omega C}}{\sqrt{R^2 + \dfrac{1}{\omega^2 C^2}}}\cos(\omega t + \phi_1) - \dfrac{R\varepsilon^{-\frac{t}{RC}}}{\sqrt{R^2 + \dfrac{1}{\omega^2 C^2}}}\sin\phi_1$

$\left. - \dfrac{\dfrac{1}{\omega C}\,\varepsilon^{-\frac{t}{RC}}}{\sqrt{R^2 + \omega^2 C^2}}\cos\phi_1\right]$

$q = \dfrac{E}{\omega\sqrt{R^2 + \dfrac{1}{\omega^2 C^2}}}\{\sin\theta\sin(\omega t + \phi_1)$

$+ \cos\theta\cos(\omega t + \phi_1) - \varepsilon^{-\frac{t}{RC}}[\sin\theta\sin\phi_1 + \cos\theta\cos\phi_1]\}$

Terms can now be combined by trigonometric identities.

(A.23) $q = \dfrac{E}{\omega\sqrt{R^2 + \dfrac{1}{\omega^2 C^2}}}[\cos(\omega t + \phi_1 - \theta)$

$- \varepsilon^{-\frac{t}{RC}}\cos(\phi_1 - \theta)]$

Substituting $\phi = \phi_1 - \theta$ in Eq. (A.23), the result corresponds to Eqs. (4.30) to (4.33) of Chap. IV.

The solution, Eq. (A.23), consists of two parts; one part varies sinusoidally with the independent variable t while the other varies exponentially with t. When the exponential term of Eq. (A.23) is substituted in the differential equation (A.03), the algebraic result is identically zero. The mathematical name of this part of the solution is *complementary function;* from the physical or engineering point of view, this is the *transient solution.* When the sinusoidal term of Eq. (A.23) is substituted in the differential equation (A.03), the algebraic result is $E\cos(\omega t + \phi_1)$. The mathematical name of this part of the solution is *particular integral;* from the physical or engineering point of view, this is the steady-state solution.

In the general solution, Eq. (A.20), there is one *arbitrary constant* $Q_{10}Q_{20} = Q_0$. This is an example of the general rule[3] that the number of arbitrary constants in the complete solution of a linear differential equation with constant coefficients is equal to the number that specifies the order of the equation.

A.03. Equations of the Second Order.—In Sec. 4.06, three second-order linear differential equations with constant coefficients are derived, and their solutions are discussed in some detail. The equations discussed in this section and correlated with the material of Chapter IV are

(A.25) $L \dfrac{d^2 q}{dt^2} + R \dfrac{dq}{dt} + \dfrac{q}{C} = E$

(A.26) $L \dfrac{d^2 q}{dt^2} + R \dfrac{dq}{dt} + \dfrac{q}{C} = E \cos (\omega t$

$$q = Q_0; \frac{dq}{dt} = Q_0'; \ t = 0$$

The solution of the homogeneous equation (A.24) is the complementary solution of Eqs. (A.25) and (A.26). Therefore the solution of Eq. (A.24) is the first subject considered in this section.

The solution of Eq. (A.24) can be approached in many ways, of which two are:

1. Guess that $A\varepsilon^{\gamma t}$ and/or $Bt\varepsilon^{\gamma t}$ are solutions; substitute and see if γ, which may be a complex number, can be evaluated in terms of L, R, C.

2. Substitute $p = d/dt$, $p^2 = d^2/dt^2$ and see what happens, treating p as an algebraic quantity until a situation arises where p must be given some other meaning in order to further the solution. Method 2 is discussed briefly in Chap. IV.

Following method 1, substitute $\varepsilon^{\gamma t}$ into Eq. (A.24)

(A.27) $$\frac{1}{C} \varepsilon^{\gamma t} = 0$$

If t does not always remain at minus infinity, γ must have a value for which

(A.28) R 1

so that

$$\gamma_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \tag{A.29}$$

or

$$\gamma_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \tag{A.30}$$

Therefore the complete solution of Eq. (A.24) is

$$q = \varepsilon^{-\frac{R}{2L}t}[A\varepsilon^{\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}t} + B\varepsilon^{-\sqrt{\frac{R}{4L^2} - \frac{1}{LC}}t}] \tag{A.31}$$

in which A and B are to be determined, in any particular problem, by two boundary conditions.

If $t\varepsilon^{\gamma t}$ is substituted in Eq. (A.24), the result is

$$\left(2\gamma + \frac{R}{L}\right) + \left(\gamma^2 + \gamma\frac{R}{L} + \frac{1}{LC}\right)t = 0 \tag{A.32}$$

This relation, Eq. (A.32), holds for all finite values of t if

$$2\gamma + \frac{R}{L} = 0 \qquad \gamma = -\frac{R}{2L} \tag{A.33}$$

and if, simultaneously

$$\gamma^2 + \frac{R}{L}\gamma + \frac{1}{LC} = 0 \tag{A.28}$$

These conditions are satisfied when

$$\frac{R}{2L} = \frac{1}{\sqrt{LC}} \tag{A.34}$$

Therefore, the solution of Eq. (A.24) can be usefully expressed for three conditions:

$$q = \varepsilon^{-\frac{R}{2L}t}(A + Bt) \qquad \text{when} \qquad \frac{R}{2L} = \frac{1}{\sqrt{LC}} \tag{A.35}$$

$$q = \varepsilon^{-\frac{R}{2L}t}[A\varepsilon^{\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}t} + B\varepsilon^{-\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}t}] \tag{A.36}$$

$$\text{when} \qquad \frac{R}{2L} > +\frac{1}{\sqrt{LC}}$$

$$(A.37) \quad q = \varepsilon^{-\frac{R}{2L}t}[A\varepsilon^{j\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}t} + B\varepsilon^{-j\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}t}]$$

$$\text{when} \quad R < +$$

These solutions of Eq. (A.24) are also the complementary solutions of Eqs. (A.25) and (A.26).

The particular solutions of Eqs. (A.25) and (A.26) can be most simply obtained by considering the physical problems which they represent. If the emf that is applied to the circuit does not vary with time, the steady-state charge q' does not vary with time either, Eq. (A.25). It follows from Eq. (A.25) that

$$(A.38) \qquad L\frac{d^2 q'}{dt^2} = 0; \qquad R\frac{dq'}{dt} = 0; \qquad \frac{q'}{C} = E$$

Therefore the particular solution of Eq. (A.25) is

$$(A.39) \qquad\qquad q' = CE$$

and the complete solution of Eq. (A.25) is the sum of Eq. (A.39) and that one of Eqs. (A.35), (A.36), (A.37) which applies in a particular case. For example, suppose that $R/2L = 1/\sqrt{LC}$ and that $q = 0$, $dq/dt = 0$ at $t = 0$. From Eqs. (A.35) and (A.39)

$$(A.40) \qquad q + q' = CE\left[1 - \left(1 + \frac{R}{2L}t\right)\varepsilon^{-\frac{R}{2L}t}\right]$$

which corresponds to Eq. (4.50).

If the emf that is applied to the circuit varies sinusoidally with time, the steady-state charge q' varies sinusoidally with t with the same frequency as the emf. Thus the particular solution q' of Eq. (A.26) is

$$(A.41) \qquad\qquad q' = Q'\cos(\omega t + \chi)$$

Substituting Eq. (A.41) in Eq. (A.26) the result is

$$(A.42) \quad -\omega^2 L Q'\cos(\omega t + \chi) - \omega R Q'\sin(\omega t + \chi) + \frac{Q'}{C}\cos(\omega t + \chi) = E\cos(\omega t + \phi_1)$$

The student can obtain values for Q' and χ by substituting $\omega t + \chi = 0$ and $\omega t + \chi = \pi/2$ in Eq. (A.42) and solving the two

equations for Q and $(\phi_1 - \chi)$.   The results are

(A.43) $Q' = -\dfrac{E}{\omega z};$ $z = \sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)}$

(A.44) $\tan(\phi_1 - \chi) = \dfrac{R}{\omega L - \dfrac{1}{\omega C}}$

Substituting these results in Eq. (A.41), the particular solution of Eq. (A.26) is

(A.45) $q' = -\dfrac{E}{\omega\sqrt{R + \left(\omega L - \dfrac{1}{\cdots C}\right)^2}} \cos\left(\omega t + \phi_1\right.$

$$\left. - \tan^{-1}\frac{R}{\omega L - \dfrac{1}{\omega C}}\right)$$

The derivative of Eq. (A.45) is precisely the steady-state current of Eq. (4.62).

It is often convenient to write the complementary solutions Eqs. (A.36) and (A.37) in terms of hyperbolic and trigonometric functions, respectively.   Since the algebra involved in these transformations is rather complicated and since most textbooks do not present the development in detail, the remainder of this section is devoted to one of these transformations.

To transform Eq. (A.36) into an equation involving hyperbolic functions, it is convenient first to designate single symbols for the fractions in the exponents of Eq. (A.36):

(A.46)   Define $\alpha = -\dfrac{R}{2r};$ $\omega = \dfrac{1}{\sqrt{LC}},$

$$\mu = \sqrt{\alpha^2 - \omega^2}$$

Equation (A.36), rewritten in terms of Eq. (A.46), is then

(A.47) $q'' = \varepsilon^{\alpha t}(A\varepsilon^{\mu t} + B\varepsilon^{-\mu t})$ $\alpha^2 > \omega^2$

writing q'' to represent only the complementary solution.   Note that α, μ, ω are all real numbers.   The first derivative of Eq. (A.47) is

(A.48) $\dfrac{dq''}{dt} = \alpha\varepsilon^{\alpha t}(A\varepsilon^{\mu t} + B\varepsilon^{-\mu t}) + \varepsilon^{\alpha t}(\mu A\varepsilon^{\mu t} - \mu B\varepsilon^{-\mu t})$

Assume that the boundary conditions are

(A.49) $q'' = 0 \cdot \qquad \dfrac{dq}{dt} = Q_0';$ at $t = 0$

so that

(A.50) $Q_0 = A + B; \qquad Q_0' = (\mu + \alpha)A \qquad (\mu - \alpha)$

Solving these equations for A and B

(A.51) $$A = \frac{Q_0}{2} + \frac{Q_0' - \alpha Q_0}{2\mu}$$

(A.52) $$B = \frac{Q_0}{2} - \frac{Q_0' - \alpha Q_0}{2\mu}$$

Now define Q

(A.53) $Q =$

Substitute Eqs. (A.51) and (A.52) into Eq. (A.47); after rearranging terms the result is

(A.54) $$q'' = \varepsilon^{\alpha t}\left[\left(\frac{Q_0'}{\mu} - \frac{\alpha Q_0}{\mu}\right)\left(\frac{\varepsilon^{\mu t} - \varepsilon^{-\mu t}}{2}\right) + Q_0\left(\frac{\varepsilon^{\mu t} + \varepsilon^{-\mu t}}{2}\right)\right]$$

This may be of the form

(A.55) $q'' = \varepsilon^{\alpha t}[(Q \sinh \delta)\,(\sinh \mu t) + Q(\cosh \delta)\,(\cosh \mu t)]$
(A.56) $q'' = Q\varepsilon^{\alpha t} \cosh (\mu t + \delta)$

If Eq. (A.56) is the correct result, the following relations must be valid

(A.57) $$\frac{Q_0'}{\mu} - \frac{\alpha Q_0}{\mu} = Q \sinh \delta$$

(A.58) $$Q_0 = Q \cosh \delta$$

These relations *are* valid because $(\cosh^2 \delta - \sinh^2 \delta)$ calculated from them and Eq. (A.53) is identically 1.   Therefore Eq. (A.36) can be written

(A.56) $q'' = Q\varepsilon^{\alpha t} \cosh (\mu t + \delta)$

and the derivative is

(A.59) $= \alpha Q\varepsilon^{\alpha t} \cosh (\mu t + \delta) + \qquad \sinh (\mu t + \delta)$

In order to derive Eq. (4.52) from these results, first note that Eq. (A.56) must take on the value $(-CE)$ at $t = 0$; this follows from the fact that q' is the particular solution of Eq. (4.46). This relation, and the boundary condition that $dq/dt = 0$ at $t = 0$, are:

(A.60) $$Q \cosh \delta = -CE$$
$$\delta) = 0$$

These relations can be solved simultaneously for $\sinh \delta$, $\cosh \delta$, and Q as follows.   From Eq. (A.61)

(A.62) $$\tanh \delta = \frac{\sinh \delta}{\cosh \delta} - \frac{\sinh \delta}{(1 + \sinh^2 \delta)^{\frac{1}{2}}} \qquad \frac{\alpha}{\mu}$$

$$1 + \sinh^2 \delta = \frac{\mu^2}{\alpha^2} \sinh^2 \delta$$

$$\sinh^2 \delta =$$

so that, writing $\sqrt{\alpha^2} = -\alpha$

(A.63) $$\sinh \delta =$$

From Eq. (A.62)

(A.64) $$\cosh \delta =$$

and from Eq. (A.60)

(A.65) $$Q = - \qquad\qquad - \alpha^2)^{\frac{1}{2}}$$

Inserting these values of $Q \sinh \delta$ and $Q \cosh \delta$ in Eq. (A.55) the result is

(A.66) $$q'' = \frac{CE}{\mu} \varepsilon^{\alpha t}(\alpha \sinh \mu t - \mu \cosh$$

which is precisely the transient part of the solution, Eq. (4.52).

The student should work out the transformation of Eq. (A.37) into the form

(A.67) $$q'' =$$

and derive Eq. (4.49) from Eq. (A.67).   This derivation can be worked out by following the procedures used to obtain Eqs. (A.56) and (A.66).

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3. *Advanced Mathematics for Engineers;* H. W. Reddick and F. H. Miller; Chap. I, John Wiley & Sons., Inc., New York, 1938.
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INTRODUCTORY NOTES ON VECTOR ANALYSIS

B.01. Introduction.—It is convenient to segregate many of the mathematical representations of physical quantities into two classes, *scalars and vectors*. A common, *though neither rigorous nor accurate*,[1] statement of the difference between these two classes is that a vector quantity has magnitude and direction while a scalar has magnitude only. The following sections are devoted to a brief list of the properties of vectors and to a few of the operations that are defined for manipulating them.

It is to be emphasized that vector analysis is a useful tool for performing operations on the symbols of certain physical quantities. The student should be constantly on guard against the tempting "explanation" that, for example, nature has provided forces that add themselves vectorially. The essential fact is that man has chosen to symbolize certain observable phenomena by some words, including among others the word "force"; and that the symbol "force" can be profitably represented by a mathematical concept—the vector. He then finds that two such vectors, each representing a force, which in turn is simply a word symbolizing an observed phenomenon, can be combined in such a way (added vectorially) that the result corresponds to some other observed phenomenon—the effect produced by the two forces acting simultaneously. Thus vector algebra is far removed from observed natural phenomena, but it has proved to be useful to represent them.

B.02. Vectors and Scalars.—Many observable physical phenomena are conveniently described by inventing certain physical quantities whose values at any point in a specified region are known or can be calculated for any instant of time in a specified interval of time. This procedure requires a means for describing unambiguously each point in the specified region and a means for specifying the value of each invented quantity at each point. These physical quantities are conveniently divided into two

classes: (1) those whose value at each point is specified by a number and a unit of measurement, and (2) those whose value at a point is specified by a number, a unit of measurement, and a direction.   The first class comprises *scalar* quantities, and members of the second class are called *vector* quantities.   A region in which a scalar or vector quantity has values at all points is commonly called a *scalar or vector field* and a mathematical relation from which such values can be calculated is called a *scalar or vector point function*.   Conventional geometric representations of vectors are described briefly below.

Let *Ox, Oy, Oz* be the three axes of a rectangular (cartesian) system of coordinates, Fig. B.01.   This particular system is
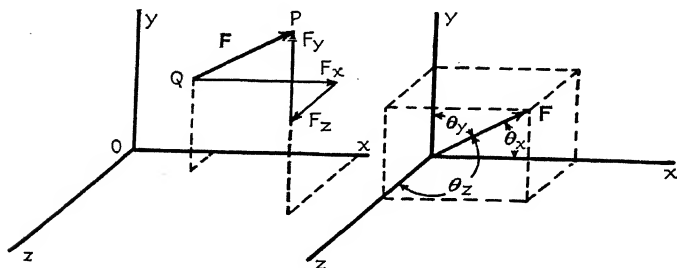
Fig. B.01.

called *right-handed* and it can be so identified by the fact that a right-hand screw pointing from *O* to *z* would advance in the positive direction along *Oz* if the slot in its head were turned in the clockwise direction from a position parallel to *Ox* to a position parallel to *Oy*.   Let i designate any straight line of unit length parallel to (or collinear with) *Ox*; let j and k designate any lines of unit length parallel to *Oy* and *Oz* respectively. These three quantities i, j, k are called *unit vectors* in the directions of the coordinate axes *Ox, Oy, Oz* respectively.   The symbols i, j, k are printed boldface to indicate that they represent numbers (in this case, 1) associated with a specified direction; *i.e.*, that they are vectors.   These unit vectors i, j, k are dimensionless.

Consider next any quantity **F** that may be specified by an expression of the form

(B.01) $$\mathbf{F} = \mathbf{i}F_x + \mathbf{j}F_y + \mathbf{k}F_z$$

where the right-hand member is taken to mean that $\mathbf{F}$ has a *magnitude*

$$(B.02) \qquad F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

and a direction that is the same as that of a straight line of length $F = \overline{QP}$ (Fig. B.01), *i.e.*, a direction determined by the direction cosines with respect to the coordinate axes

$$(B.03) \quad \cos\theta_x = \frac{F_x}{F}; \qquad\qquad\qquad \cos\theta_z = \frac{F_z}{F}$$

If these relations hold irrespective of the choice of axes, and if the quantity $\mathbf{F}$ is such that, in relation to similar quantities, it obeys certain rules of addition and multiplication described below, the quantity $\mathbf{F}$ is called a *three-dimensional space vector* or, more commonly, simply a *vector*.* The vectors $\mathbf{i}F_x$, $\mathbf{j}F_y$, $\mathbf{k}F_z$ (taken as vectors having directions parallel to Ox, Oy, Oz, and having magnitudes F_x, F_y, F_z respectively) are called the *components* of the vector $\mathbf{F}$ in the directions of the coordinate axes. The numbers F_x, F_y, F_z, having according to this notation magnitude but not direction, are called the *scalar values* of the components of the vector $\mathbf{F}$.

The simplest vector is a directed line of specified length. Displacement, velocity, acceleration, and force can be represented by vectors. The flow of something per unit area, such as water, through a small area perpendicular to the direction of flow at a point in the area, is commonly called the *density* of flow and is commonly represented by a vector. The electric-current density, electric-field intensity, electric-flux density, magnetic-field intensity, and magnetic-flux density are vector quantities discussed in this book, particularly in Part 2.

As noted above, a quantity that may be completely specified by a positive or negative number and a unit of measurement, but that has no direction in space, is called a *scalar quantity*. Mass, temperature, mass density, work, energy, electric charge, volume density or surface density of electric charge, electric current, electric flux, and magnetic flux are scalar quantities. Note

* The concept of a vector may be extended to space of any number of dimensions, *i.e.*, to quantities that are functions of any number of independent variables, but this more general concept is not considered in this book.[1]

that, although current and flux are scalar quantities, current *density* and flux *density* are vector quantities.

B.03. Definition of the Product of a Vector by a Scalar.— Designate by **G** any vector whose components in the directions of the three axes Ox, Oy, Oz have the magnitudes

$$(B.04) \qquad \begin{aligned} G_x &= uF_x \\ G_y &= uF_y \\ G_z &= uF_z \end{aligned}$$

where u is any specified *scalar*.

From the defining equations (B.01) to Eq. (B.03) the vector **G** is a vector parallel to **F** and with a magnitude equal to the product of u and the magnitude of **F**, *viz.*,

$$(B.05) \qquad G = uF = Fu$$

From Eq. (B.01) the vector **G** may then be written

$$(B.06) \qquad G = u(iF_x + jF_y + kF_z) = u\mathbf{F} = \mathbf{F}u$$

The vector $\mathbf{G} = u\mathbf{F} = \mathbf{F}u$ is not, strictly speaking, a "product" in the ordinary algebraic sense; however, it is convenient to refer to $u\mathbf{F}$ or $\mathbf{F}u$ as the *product of a vector by a scalar*. Note that the commutative law of algebra here holds and that the vector $u\mathbf{F} = \mathbf{F}u$ is always *parallel* to or collinear with the vector **F**.

B.04. Definitions of the Sum and Difference of Two Vectors.— The vector whose components in the direction of the three axes Ox, Oy, and Oz have respectively the magnitudes

$$(B.07) \qquad \begin{aligned} &= F_x + G_x \\ S_y &= F_y + G_y \\ &= F_z + G_z \end{aligned}$$

where F_x, F_y, and F_z are the magnitudes of the corresponding components of a vector **F**, and G_x, G_y, G_z are the magnitudes of the corresponding components of a vector **G**, is defined as the *vector sum*, or simple the *sum*, of the two vectors **F** and **G**. This concept of a vector sum may be extended to any number of vectors, the basic idea being that one adds *algebraically* only components that are in the *same direction*. The sum of two vectors is illustrated in Fig. B.02. The sum of any number of

vectors is usually expressed by the single relation

(B.08) $S = F + G + H + \cdots$

it being understood that the plus signs here mean vector addition as just defined.

In exactly the same manner, the vector whose components in the directions of the three axes have respectively the magnitudes

(B.09)
$$D_x = F_x - G_x$$
$$D_y = F_y - G_y$$

is defined as the *vector difference*, or simply the *difference*, of the two vectors **F** and **G**, with **F** as the pre-term (or minuend).   The
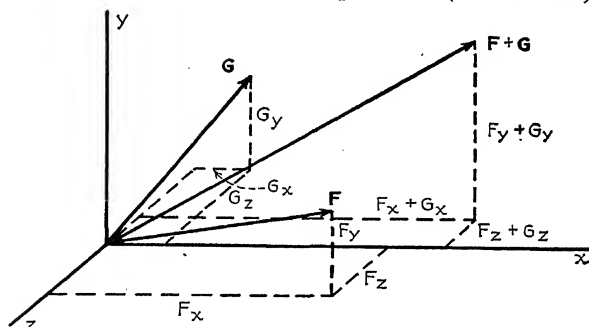
Fig. B.02.

difference of any two vectors **F** and **G**, with **F** as the pre-term, is usually expressed by the single formula

(B.10) $\mathbf{D} = \mathbf{F} - \mathbf{G}$

it being understood that the minus sign here means *vector subtraction* as just defined.

By the *negative of a vector* **F** is meant the difference between this vector and a vector of zero magnitude as the prefactor, *viz.*,

(B.11) $-\mathbf{F} = \mathbf{0} - \mathbf{F}$

Such a vector has the same magnitude as the vector F but is in the *opposite* direction in space.

B.05. Definition of the Scalar Product of Two Vectors.—The product of *two* vectors, in the ordinary algebraic sense, has no meaning, any more than the product of one barrel of apples by

another barrel of apples has an algebraic meaning.   However, it is convenient to *define arbitrarily* two kinds of vector products, called respectively their *scalar product* and their *vector product*. In terms of each of these products may be expressed certain important physical facts, as will become apparent immediately.

The reader is familiar with the expression for the work W done by a force of magnitude F on a particle that is displaced a distance x under the action of this force:

$$(B.12) \qquad W = (F \cos \theta)x$$

where θ is the angle between the direction of the force and the direction of the displacement.   A much more convenient way of writing this expression is

$$(B.13) \qquad W = \mathbf{F} \cdot \mathbf{x}$$

where $\mathbf{F}$ is the *vector* that has the magnitude and direction of the force and $\mathbf{x}$ is the *vector* that has the magnitude and direction of the displacement.   The quantity $\mathbf{F} \cdot \mathbf{x}$ is *defined* as the *scalar product* of the two vectors $\mathbf{F}$ and $\mathbf{x}$; see Fig. B.03.
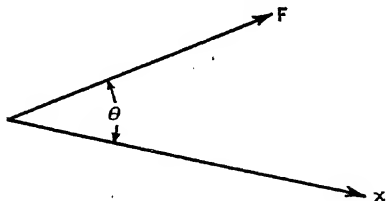
Fig. B.03.

In general, the scalar product of any two vectors $\mathbf{F}$ and $\mathbf{G}$, irrespective of what physical quantities they may represent, is *defined* as the *scalar FG* $\cos \theta$ where θ is the angle between the directions of $\mathbf{F}$ and $\mathbf{G}$

$$(B.14) \qquad \mathbf{F} \cdot \mathbf{G} = \mathbf{G} \cdot \mathbf{F} = FG \cos \theta$$

Note that the sequence in which the two factors $\mathbf{F}$ and $\mathbf{G}$ are written is immaterial, *i.e.*, in the scalar product of two vectors, the commutative law of algebra holds.

From the defining equation (B.14) it follows that the scalar product of two *parallel* vectors ($\theta = 0$) is equal to the algebraic

product of their magnitudes

(B.15) $\mathbf{F} \cdot \mathbf{G} = FG$ for $\theta = 0$

and that the scalar product of two mutually *perpendicular* vectors ($\theta = 90$ degrees) is zero

(B.16) $\mathbf{F} \cdot \mathbf{G} = 0$ for $\theta = 90$ degrees

As a consequence of these relations it follows that, for the three mutually perpendicular *unit vectors* $\mathbf{i}, \mathbf{j}, \mathbf{k}$:

(B.17)
$$\begin{aligned}
\mathbf{i} \cdot \mathbf{i} &= 1 & \mathbf{i} \cdot \mathbf{j} &= \mathbf{j} \cdot \mathbf{i} = 0 \\
\mathbf{j} \cdot \mathbf{j} &= 1 & \mathbf{j} \cdot \mathbf{k} &= \mathbf{k} \cdot \mathbf{j} = 0 \\
\mathbf{k} \cdot \mathbf{k} &= 1 & \mathbf{k} \cdot \mathbf{i} &= \mathbf{i} \cdot \mathbf{k} = 0
\end{aligned}$$

Consequently, when two vectors $\mathbf{F}$ and $\mathbf{G}$ are each expressed as a vector sum of their three mutually perpendicular components

(B.18)
$$\begin{aligned}
\mathbf{F} &= \mathbf{i}F_x + \mathbf{j}F_y + \mathbf{k}F_z \\
\mathbf{G} &= \mathbf{i}G_x + \mathbf{j}G_y + \mathbf{k}G_z
\end{aligned}$$

the scalar product of $\mathbf{F}$ and $\mathbf{G}$ may be found by multiplying these two quantities by the ordinary algebraic law for the product of two polynomials, placing dots between the unit vectors to indicate scalar products and then substituting for the scalar products of the unit vectors the values given by Eq. (B.17). There results

(B.19) $\mathbf{F} \cdot \mathbf{G} = \mathbf{G} \cdot \mathbf{F} = F_x G_x + F_y G_y + F_z G_z$

The scalar product is sometimes called the *dot product* or the *inner product*.

B.06. Definition of the Vector Product of Two Vectors.—The second type of product of two vectors $\mathbf{F}$ and $\mathbf{G}$, and one which has numerous useful applications, is the *vector* whose magnitude is equal to the triple product of the magnitudes of $\mathbf{F}$ and $\mathbf{G}$ and the modulus (numerical value without sign) of the *sine* of the angle between them, and whose direction is *perpendicular* to both $\mathbf{F}$ and $\mathbf{G}$. Such a "product" is called the *vector product* of the two vectors $\mathbf{F}$ and $\mathbf{G}$. In other words,

(B.20)   Magnitude of cross product of $\mathbf{F}$ by $\mathbf{G} = FG \, |\sin \theta|$

and the direction of this cross product is perpendicular to the plane determined by two intersecting lines parallel respectively

to **F** and **G**. The *positive sense* of this cross-product vector is arbitrarily defined as the sense in which a right-handed screw advances, when given not more than *half* a turn in such a direction as to make a radial line in its head, having the same direction outward from the center of the head as the direction of the *prefactor* **F**, take the direction of the *postfactor* **G**. For example, if in Fig. B.04 the two vectors **F** and **G** are in the plane of the paper, and θ is less than 180 degrees, then the vector product of **F** and **G** is perpendicular to the plane of the paper in the direction *away from* the eye of the reader:
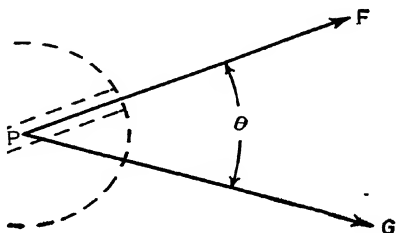
Fig. B.04.

The generally adopted symbol for a vector product is the composite symbol formed by placing a cross between the prefactor and the postfactor

(B.21) **F** $\times$ **G** $\equiv$ vector product of **F** and **G** with **F** as the
prefactor

From the convention in regard to the direction of this vector, it follows that the vector product of **F** and **G**, with **G** as the prefactor is

(B.21) $$\mathbf{G} \times \mathbf{F} = -\mathbf{F} \times \mathbf{G}$$

From the definition of the vector product it follows that the vector product of pairs of three mutually perpendicular *unit vectors* are

(B.22)
$$\begin{aligned}
\mathbf{i} \times \mathbf{i} &= 0 & \mathbf{i} \times \mathbf{j} &= -\mathbf{j} \times \mathbf{i} = \mathbf{k} \\
\mathbf{j} \times \mathbf{j} &= 0 & \mathbf{j} \times \mathbf{k} &= -\mathbf{k} \times \mathbf{j} = \mathbf{i} \\
\mathbf{k} \times \mathbf{k} &= 0 & \mathbf{k} \times \mathbf{i} &= -\mathbf{i} \times \mathbf{k} = \mathbf{j}
\end{aligned}$$

From these relations it follows that when **F** and **G** are expressed in terms of their components in the directions of three mutually

perpendicular axes, as in Eq. (B.18), the vector product $\mathbf{F} \times \mathbf{G}$ may be found by multiplying these two quantities by the ordinary algebraic law for the product of two polynomials, keeping the components of $\mathbf{F}$ always as *prefactors* and placing a cross between the unit vectors to indicate vector products and then substituting for the vector products of the unit vectors the values given by Eq. (B.22). There results

(B.23) $\mathbf{F} \times \mathbf{G} = \mathbf{i}(F_y G_z - F_z G_y) + \mathbf{j}(F_z G_x -$

Note the cyclical sequence of the subscripts in the three components of this vector product.
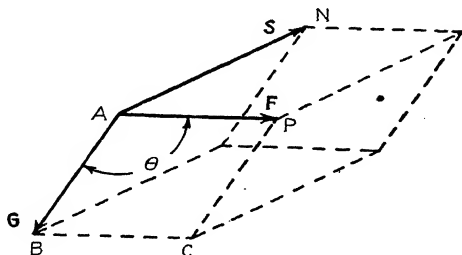
FIG. B.05.

A simple example of a vector product is a so-called *directed area*. By such an area is meant a vector whose magnitude is this *area* and whose direction is the *normal* to this area in a specified sense. For example, consider a plane parallelogram whose adjacent edges are two straight lines $\mathbf{F}$ and $\mathbf{G}$, the first drawn toward a corner P and the other drawn toward a corner B, as in Fig. B.05, and let θ be the angle, taken less than 180 degrees, between their directions. Then the area of the parallelogram is $FG \sin \theta$, where F and G are the lengths of the two lines $\mathbf{F}$ and $\mathbf{G}$. The direction of this plane surface relative to a specified set of coordinate axes is completely determined by the direction of the line AN, normal to this surface, drawn in the right-handed screw direction with respect to the circuital sequence of the arrows that represent the positive senses of $\mathbf{F}$ and $\mathbf{G}$. The parallelogram $APCB$ when considered as a *directed area*, is then the vector

(B.24) $\mathbf{S} = \mathbf{F} \times \mathbf{G}$

This vector has a magnitude equal to the area of the parallelogram and the direction of the normal AN. Note that this direction is also that which bears a right-handed screw relation to the motion of a point that traces the boundary of this area in the clockwise direction when viewed from the side of the surface *opposite* to that *from* which the normal AN is drawn.

From the above definitions of scalar product and vector product the following relations for the triple products of three vectors **F, G, H** are readily deduced

(B.25) $\mathbf{F} \cdot (\mathbf{G} \times \mathbf{H}) = (\mathbf{F} \times \mathbf{G}) \cdot \mathbf{H} = \mathbf{G} \cdot (\mathbf{H} \times \mathbf{F})$

(B.26) $\mathbf{F} \times (\mathbf{G} \times \mathbf{H}) = (\mathbf{F} \cdot \mathbf{H})\mathbf{G} - (\mathbf{F} \cdot \mathbf{G})\mathbf{H}$

The reader's attention is called to the fact that $(\mathbf{F} \times \mathbf{G}) \cdot \mathbf{H}$ is *not* equal to $\mathbf{F} \times (\mathbf{G} \cdot \mathbf{H})$, since $(\mathbf{G} \cdot \mathbf{H})$ is a scalar, and the vector product of a *scalar* by a vector has no meaning.

The vector product is sometimes called the *cross product* or *outer product*. *Note that vector division is not defined.*

B.07. Definition of the Flux of a Vector.—The chief economy of the vectorial representation of physical quantities is based upon the symbolizing of a relatively complicated operation on one or more vectors by a simple and brief name, such as vector product, divergence, gradient, curl, and circulation. This is also the chief source of confusion for all who attempt for the first time to use vectors to represent physical quantities. The remainder of this appendix is devoted to a description of a few of the vector operations commonly used for the vector analysis of physical quantities.

A region throughout which a scalar or vector quantity is single-valued and continuous is, as noted above, commonly called a *scalar* or *vector field*. In a vector field it is often useful to add algebraically over a specified surface S the scalar products of the form

$$\mathbf{F} \cdot \mathbf{dS}$$

in which **F** is the vector of the field under consideration and **dS** is the vector representing an infinitesimal part of the surface S. It will be recalled that **dS** is assigned the magnitude dS and the direction perpendicular to dS *in a specified sense* (Sec. B.06). This summation is a kind of integration, subject in many cases to rules of integral calculus. The result of the operation is

commonly called the *flux* ϕ of the vector **F** through the surface
S in the specified sense:

(B.27)

B.08. The Definition of the Line Integral of a Vector; Circulation.—The scalar product of a vector **F** and a differential length
ds is

(B.28) $\mathbf{F} \cdot \mathbf{ds} = F_x\,dx + F_y\,dy + F_z\,dz$

The *line integral* of a vector along a specified path in a specified
sense from a point A to a point B in a vector field is often a useful
quantity from the physical point of view.

The line integral of a vector **F** about a specified closed path
in the field of **F** is written

(B.29) $$\oint \mathbf{F} \cdot \mathbf{ds}$$

and is called the *circulation* of the vector **F** about the specified
path *in the specified sense.*   If

(B.30) $$\oint \mathbf{F} \cdot \mathbf{ds} = 0$$

for all possible paths in a region, the vector **F** is said to be an
irrotational vector in that region.

B.09. The Definition of the Derivative of a Vector.—If the
magnitude and/or direction of a quantity represented by a
vector changes when an independent variable t changes, the
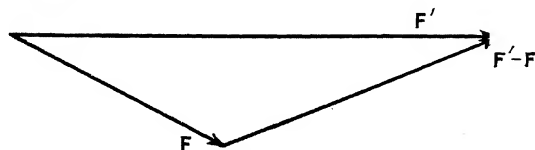
Fig. B.06.

vector may be designated **F** and **F'** for t and $t + \Delta t$ respectively.
The derivative of **F** with respect to t is defined as

(B.31) $$\frac{d\mathbf{F}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{F'} - \mathbf{F}}{\Delta t}$$

This quantity is a vector.   If the *direction* of the vector is inde-
pendent of t, the derivative is collinear with **F**; if the *magnitude*

of the vector is independent of t, the derivative is perpendicular to $\mathbf{F}$. A general example is shown in Fig. B.06.

B.10. The Definition of the Gradient of a Scalar.—A scalar field, described in Sec. B.02, can be represented in a right-hand cartesian system of coordinates by specifying the value of the scalar V as an algebraic function of the coordinates

$$(\text{B.32}) \qquad V = f(x, y, z)$$

In many cases in physics the quantities representable by scalars are not only continuous and single-valued, but their first and second partial derivatives with respect to the coordinates x, y, z, are similarly "well behaved." In such cases certain useful vectors can be defined in terms of the scalar V and the coordinate system.

Note that in general the differential of V is

$$(\text{B.33}) \qquad dV = \frac{\partial V}{\partial x}\, dx + \frac{\partial V}{\partial y}\, dy + \frac{\partial V}{\partial z}\, dz$$

plus other terms involving second or higher order differentials, which are ignored. Note also that the scalar product of a vector $\mathbf{P}$ and a differential element of length $\mathbf{ds} = \mathbf{i}\, dx + \mathbf{j}\, dy + \mathbf{k}\, dz$ is

$$(\text{B.34}) \qquad \mathbf{P} \cdot \mathbf{ds} = P_x\, dx + P_y\, d$$

It follows from Eqs. (B.33) and (B.34) that if

(B.35)

then

$$(\text{B.36}) \qquad \int_A^B \mathbf{P} \cdot \mathbf{ds} = V_B - V_A$$

The $\mathbf{P}$ so chosen is called the *gradient of the scalar* V, so that Eq. (B.35) is the definition of the vector* **grad** V. It is clear that the circulation of **grad** V is zero.

$$(\text{B.37}) \qquad \oint \mathbf{grad}\ V \cdot \mathbf{ds} = 0$$

Thus an irrotational vector can be expressed as the gradient of a scalar.

---

* The more common notation ∇V for the gradient of V is discussed in Sec. B.13, together with other uses of the vector operator ∇ (del), which is there defined.

The magnitude of **grad** V at a point in the scalar field is

$$(\text{B.38}) \qquad \text{grad } V = \left[\left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 + \left(\frac{\partial V}{\partial z} \right)^2 \right]^{\frac{1}{2}}$$

and the direction of **grad** V, can be specified in terms of the angles between **grad** V and the axes

$$\cos \theta_x = \frac{\dfrac{\partial V}{\partial x}}{\left[\left(\dfrac{\partial V}{\partial x} \right)^2 + \left(\dfrac{\partial V}{\partial y} \right)^2 + \left(\dfrac{\partial V}{\partial z} \right)^2 \right]^{\frac{1}{2}}}$$

$$(\text{B.38}a) \qquad \cos \theta_y = \frac{\dfrac{\partial V}{\partial y}}{\left[\left(\dfrac{\partial V}{\partial x} \right)^2 + \left(\dfrac{\partial V}{\partial y} \right)^2 + \left(\dfrac{\partial V}{\partial z} \right)^2 \right]^{\frac{1}{2}}}$$

$$\cos \theta_z = \frac{\dfrac{\partial V}{\partial z}}{\left[\left(\dfrac{\partial V}{\partial x} \right)^2 + \left(\dfrac{\partial V}{\partial y} \right)^2 + \left(\dfrac{\partial V}{\partial z} \right)^2 \right]^{\frac{1}{2}}}$$

Thus the rate of change $\partial V / \partial \alpha$ in any direction α perpendicular to **grad** V is zero. Therefore there are surfaces in the scalar field of V over which the value of V is constant. Such surfaces are called *equipotential surfaces* of the scalar field. *The vector* **grad** V *is always perpendicular to such surfaces.*

B.11. The Definition of the Divergence of a Vector.—Figure B.07 shows an enlarged view of an imaginary cube about the point P in a vector field. The net outward flux of the vector **F** from this cube can be calculated from the application of Eq. (B.27), assuming that the vector is constant to within a second-order differential over each face of the cube. If the components of **F** at P are F_x, F_y, F_z, the flux of **F** *outward* through the face $abcd$ is

$$\left(F_x \quad \frac{\partial F_x}{\partial x} \frac{\Delta x}{2} \right) \Delta y \, \Delta z$$

and the *outward* flux through the face $a'b'c'd'$ is

$$- \left(F_x - \frac{\partial F_x}{\partial x} \frac{\Delta x}{2} \right) \Delta y \, \Delta z$$

The net outward flux $\Delta\phi$ through all six sides of the cube, calculated in this manner, is

(B.39) $\Delta\phi =$

The net outward flux *per unit volume* is called the divergence of the vector **F**

(B.40) $\operatorname{div} \mathbf{F} = \dfrac{\partial F_x}{\partial x} \quad \partial y \quad \partial z$ (a scalar)

It follows from Eq. (B.39) that the total outward flux ϕ of the
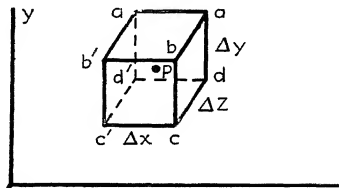
FIG. B.07.

vector **F** through a closed surface is

(B.41) $\phi =$

and from Eq. (B.27)

(B.42) $\displaystyle\int_S \mathbf{F} \cdot \mathbf{dS} = \int_v \operatorname{div} \mathbf{F} \, dv$

This relation is called *Gauss's theorem.*

   Note that a vector whose divergence is zero is called a *solenoidal vector.*

   B.12. Definition of the Curl of a Vector.—The curl of a vector **F** is a vector; the magnitude of **curl F** and its direction are defined as follows. Suppose that Fig. B.08 represents a small part of the vector field of a vector **F**, and that in the neighborhood

of the point P, the vector **F** lies in the plane AB.   In other words, the small surface ΔS is coplanar with **F**.   The magnitude of **curl F** at the point P is defined as limit of the ratio of the circulation of **F** about the path s in the direction indicated in Fig. B.08, to the area ΔS, as ΔS approaches zero; the direction of **curl F** at the point P is the direction perpendicular to the surface ΔS in the sense indicated in Fig. B.08.   Note that the positive direction of **curl F** and the direction of "integration" of **F** about the path s bear to each other a right–hand–screw relationship.
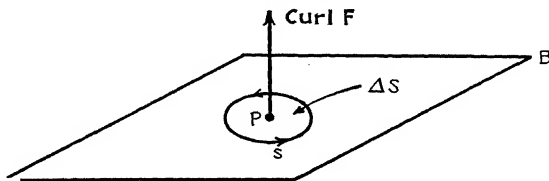
FIG. B.08.
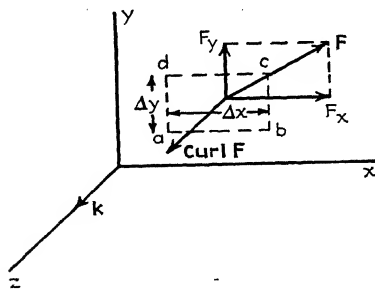
FIG. B.09.

A simple example of the curl of a vector can be worked out from Fig. B.09.   In this example **F** in the neighborhood of the point P is in the xy plane.   Therefore **F** has only two components F_x and F_y; and according to the definition of **curl F**, *it* has only one component, parallel to the z axis as indicated in B.08.

The magnitude of **curl F** for the simple example shown in B.09 is calculated as follows.   The circulation of **F** about the rectangle in Fig. B.09 *in the counterclockwise direction* is the sum of four terms:

from a to b $\qquad F_x - \dfrac{\partial F_x}{\partial y} \dfrac{\Delta y}{2}$

from b to c

from c to d $\qquad -\left(F_x + \dfrac{\partial F_x}{\partial y} \dfrac{\Delta y}{2} \right) \Delta x$

from d to a

and this sum is

Therefore, the curl of the vector $\mathbf{F}$ is, in this particular example,

(B.43)

In general, the curl of a vector has three components. In rectangular coordinates, the curl of a vector is

An extremely useful integral theorem can be deduced from the definition of the curl of a vector. This theorem is called Stokes's theorem; it states that the circulation of a vector $\mathbf{F}$ around a closed loop s is equal to the surface integral of **curl F** over any open surface bounded by the closed loop s. The theorem can be inferred from the following discussion.

Figure B.10 shows a closed loop s and an open surface S. The surface can be considered as being made up of many four-sided areas ΔS as indicated over part of the surface in Fig. B.10. The curl of the vector $\mathbf{F}$, at the center of a particular ΔS such as that marked a in Fig. B.10, may have a direction different from the vector $\mathbf{dS}$; both **curl F** and **dS** being limits as ΔS approaches dS. Note the following relations, for the area a:

(B.44) $\qquad \lim \dfrac{\displaystyle\oint_a \mathbf{F} \cdot \mathbf{ds}}{\Delta S} \qquad\qquad \cos\theta$

(B.45) $$\lim_{\Delta S \to dS} \oint_a \mathbf{F} \cdot \mathbf{ds} = \text{curl } \mathbf{F} \cdot \mathbf{dS}$$

Now if an "equation" like Eq. (B.45) is set up for each four-side area ΔS, and the results added, the sum of the right-hand members is

$$\int_S \text{curl } \mathbf{F} \cdot \mathbf{dS}$$

When the left-hand members of equations like Eq. (B.45) are combined, each side of every *interior* four-sided figure takes part twice, once in one direction, and once in the other, the *net result*
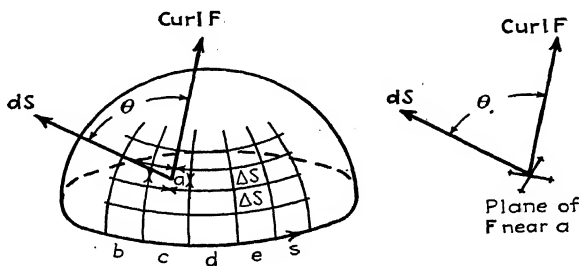
FIG. B.10.

being zero. This is indicated by arrows in Fig. B.10. Only the contributions of segments such as b, c, d, e on the loop s remain as the "line integrals" are combined. Therefore the combination of all the left-hand members of "equations" like Eq. (B.45) is

$$\mathbf{F} \cdot \mathbf{ds}$$

so that

(B.46) $$\oint \mathbf{F} \cdot \mathbf{ds} = \int_S \text{curl } \mathbf{F} \cdot \mathbf{dS}$$ (Stokes's theorem)

Note that, if curl $\mathbf{F} = 0$, $\mathbf{F}$ is said to be an *irrotational* vector.

B.13. Definition of the Vector Operator Nabla (Del).—It was shown in Secs. B.05 and B.06 that the scalar and vector products of two vectors can be calculated by expressing the two vectors in terms of their components and multiplying term by term:

$$\mathbf{F} \cdot \mathbf{G} = (\mathbf{i}F_x + \mathbf{j}F_y + \quad (\mathbf{i}G_x + \mathbf{j}G_y + \mathbf{k}G_z)$$
$$\mathbf{F} \times \mathbf{G} = (\mathbf{i}F_x + \mathbf{j}F_y + \quad \times$$

The gradient, divergence, and curl can be calculated in an analogous manner by means of a *vector operator nabla* or *del* whose symbol is ∇

$$\text{(B.47)} \qquad\qquad \mathbf{i}\,\frac{\partial}{\_} + \mathbf{j}\ ^{\partial}$$

This symbol is both a vector and a differential operator. If it is prefixed to a scalar V, the result is

$$\text{(B.48)} \qquad\qquad \nabla V = \mathbf{i}\,\frac{\partial V}{\partial x} + \mathbf{j}\,\frac{\partial V}{\partial y} + \mathbf{k}\,\frac{\partial V}{\partial z}$$

which is **grad** V. Similarly

$$\text{(B.49)} \qquad \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \mathbf{div}\ \mathbf{F}$$

$$\frac{\partial r\,_y}{\_} - \frac{\partial r\,_x}{\_}\Big) = \mathbf{curl}\ \mathbf{F}$$

(B.51)

The scalar operator $\nabla \cdot \nabla$ is often written ∇^2; it is called the *Laplacian operator*. The student should check the following identities by using the operator ∇, Eq. (B.47), in conjunction with the definitions of Secs. B.02 to B.12:

$$\text{(B.52)} \qquad\qquad \mathbf{div\ grad}\ V = \nabla^2 V$$
$$\text{(B.53)} \qquad\quad \nabla \cdot (\nabla \times \mathbf{F}) = \mathbf{div\ curl}\ \mathbf{F} = 0$$
$$\text{(B.54)} \qquad\quad \nabla \times (\nabla V) = \mathbf{curl\ (grad}\ V) = 0$$
$$\text{(B.55)} \quad \nabla \times \nabla \times \mathbf{F} = \mathbf{curl\ curl}\ \mathbf{F} = \mathbf{grad\ div}\ \mathbf{F} \qquad \nabla^2 \mathbf{F}$$

The symbol $\nabla^2 \mathbf{F}$ represents

$$\text{(B.56)} \qquad\qquad \nabla^2 \mathbf{F} =$$

in which $\nabla^2 F_x$, $\nabla^2 F_y$, $\nabla^2 F_z$ are of the form of Eq. (B.51).

B.14. Conclusion.—The important relations among vector quantities and operations upon them have been presented here briefly so that the chapters of Part 2 need not be complicated by developing them there. For a rigorous mathematical derivation of these quantities, the student is referred to standard

works on vector analysis.[2,3] The relations given here for right-handed cartesian coordinates can also be expressed in terms of cylindrical and spherical coordinates. These formulas are derived in Appendix C.

References

1. *Vector and Tensor Analysis;* Homer V. Craig; McGraw-Hill Book Company, Inc., New York, 1943.
2. *Vector Analysis;* J. W. Gibbs; Yale University Press, New Haven, 1931.
3. *Vector Analysis;* A. P. Wills; Prentice-Hall, Inc., New York, 1931.

Problems

B.01. Prove that $A = i9 + j - k6$ and $B = i4 - j6 + k5$ are perpendicular to each other.

B.02. Calculate the smaller angle between $A = i3 + j4 + k2$ and $B = i2 + j3 + k4$.

B.03. Show that the angle ϕ between two vectors is the angle whose cosine is a function of the cosines of the angles between each vector and the three coordinate axes, and state in terms of these six cosines the conditions for the vectors' being perpendicular and parallel to each other.

B.04. Prove that the diagonals of a parallelogram bisect each other by a vector method.

B.05. Prove that $\nabla^2(1/r) = 0$ except at $r = 0$ $(r^2 = x^2 + y^2 + x^2)$.

B.06. Calculate $\sin (x - y)$ as a function of the sines and cosines of x and y by vector methods.

B.07. Show that if $(xyz)^b(ix^a + jy^a + kz^a)$ is irrotational, either $b = 0$ or $a = -1$.

B.08. Prove that $\int_S A \cdot dS = \frac{12}{5}\pi R^2$ in which R is the radius of a sphere of surface S and $A = ix^3 + jy^3 + kz^3$.

B.09. Prove that if r_1 is a unit vector and $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, then $r^{a+1}r_1$ is solenoidal if $a = -3$, but irrotational for all other values of a.

B.10. Show that $\nabla \cdot (aA) = a\nabla \cdot A + A \cdot \nabla a$.

B.11. Show that $\nabla \times (aA) = a\nabla \times A - A \times (\nabla a)$.

B.12. Show that $\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$.

B.13. The vector α is perpendicular to a plane containing a radius vector r. The magnitude of α is: $\alpha = \cos \omega t$ for $0 \leq r \leq R$; $\alpha = 0$ for $r > R$. A second vector β is related to α by $\nabla \times \beta = -\partial\alpha/\partial t$. Calculate β in the plane for $r > R$. Discuss in detail.

APPENDIX C

VECTOR FORMS

IN ORTHOGONAL CURVILINEAR COORDINATES

It is the purpose of this appendix to show how to calculate ∇V, $\nabla \cdot \mathbf{A}$, $\nabla \times \mathbf{A}$, and $\nabla^2 V$ in general orthogonal curvilinear coordinates and to derive these functions in spherical coordinates (r, θ, ϕ) and in cylindrical coordinates (r, ϕ, z). Orthogonal curvilinear coordinates u, v, w are space coordinates, each expressible as a function of x, y, z, such that

$$u(x, y, z) = c_1$$

$$w(x, y, z) = c_3$$

are mutually perpendicular surfaces near $x = x_1$, y $\quad y_1$, $z =$ the point x_1, y_1, z_1 being any point in that region where there are no discontinuities in u, v w or in their derivatives.
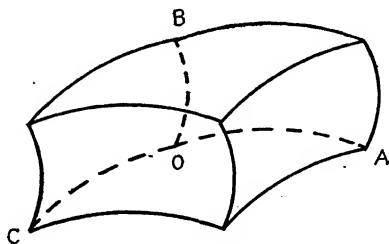
FIG. C.01.

The differentials du, dv, dw do not in general represent lineal differential elements; for example, if $du = d\theta$, where θ is an angle, the lineal differential element corresponding to $d\theta$ may be $r d\theta$, where r is a length. In general, the lineal differential elements corresponding to du, dv, dw can be written: $\alpha \, du$, $\beta \, dv$, $\gamma \, dw$, where α, β, γ may be functions of u, v, w and must be so treated. The quantities $\mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1$ are unit vectors; at a particular point, they are mutually perpendicular, and tangent to the surfaces $u = c_1, v = c_2, w = c_3$ discussed above. In Fig. C.01, the "solid" represents a

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differential volume in u, v, w space. Note the following relations, using $\rightarrow$ to mean "corresponds to":

$$OA \rightarrow du$$
$$OB \rightarrow dv$$
$$OC \rightarrow dw$$

Thus for convenience u, v, w is taken as the cyclic order; this makes the new system a right-hand system.

In the following paragraphs the functions ∇V, $\nabla \cdot \mathbf{A}$, $\nabla \times \mathbf{A}$, $\nabla^2 V$ are derived in the same manner as they were derived for rectangular coordinates. In each case the resulting orthogonal curvilinear coordinate expression is reduced to the expressions for cylindrical and spherical coordinates; these systems are illustrated in Figs. C.02 and C.03.

u ⟶ r
v ⟶ φ
w ⟶ z
α ⟶ 1
β ⟶ r
γ ⟶ 1

dz r dφ
P dr
z

Fɪɢ. C.02.—Cylindrical coordinates r, ϕ, z.

u ⟶ r
v ⟶ θ
w ⟶ φ
α ⟶ 1
β ⟶ r
γ ⟶ r sin θ

r sin θ dφ
dr
θ r
r dθ
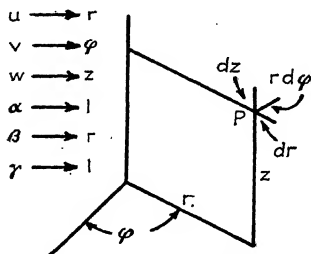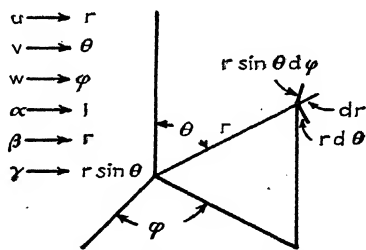
Fɪɢ. C.03.—Spherical coordinates r, Θ, ϕ.

If $V(x, y, z)$ is a continuous, single-valued function with continuous, single-valued first and second partial derivatives it is a suitable candidate for being the potential of a vector $\mathbf{A}$:

(C.02)

In general orthogonal curvilinear coordinates u, v, w, there are three components of **grad** V at a point P; each is the limit of the change in V corresponding to a lineal measure of distance from P in the direction tangent to the increase of a coordinate as the distance approaches zero, for example:

$$\mathbf{grad}\ V\Big]_u = \frac{1}{\alpha} \frac{\partial V}{\partial u}$$

Therefore

(C.03) $\nabla V =$ $r = \dfrac{1}{\alpha}\dfrac{\partial V}{\partial u}\,\mathbf{u_1} + \dfrac{1}{\beta}\dfrac{\partial V}{\partial v}\,\mathbf{v_1} + \dfrac{1}{\gamma}\dfrac{\partial V}{\partial w}\,\mathbf{w_1}$

(general orthogonal curvilinear coordinates)

Substituting the corresponding quantities from Figs. C.02 and C.03:

(C.04)

(cylindrical coordinates)

(C.05)

(spherical coordinates)

In order to calculate the divergence of a vector $\mathbf{A}$ in curvilinear coordinates, the outward flux of $\mathbf{A}$ from an infinitesimal volume expressed in terms of these coordinates divided by this volume
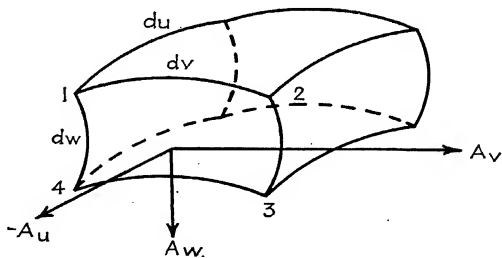
FIG. C.04.

is calculated. Suppose that the average value of the vector $\mathbf{A}$ over the nearest face of the infinitesimal volume of Fig. C.04 is $-A_u$; the outward flux from the nearer face is

$$-A_u\beta\,dv\gamma\,dw$$

because the area through which $-A_u$ is the normal component of $\mathbf{A}$ is $(\beta\,dv)(\gamma\,dw)$. The area and the normal component of $\mathbf{A}$ for the opposite surface are both changed, so that the flux of $\mathbf{A}$ outward from the opposite face is

$$_u\beta\,dv\gamma\,dw + \dfrac{\partial}{\partial u}\,(A_u\beta\,dv\gamma\,dw)\,du$$

Therefore the net outward flux of **A** through these two surfaces is

$$\frac{\partial}{\partial u} (\beta\gamma A_u) \, du \, dv \, dw$$

From symmetry the other two contributions to the outward flux of **A** are

$$\frac{\partial}{\partial v} (\gamma\alpha A_v) \, du \, dv \, dw$$

$$\frac{\partial}{\partial w} (\alpha\beta A_w) \, du \, dv \, dw$$

Therefore the outward flux of **A** *per unit volume* $(\alpha \, du\beta \, dv\gamma \, dw)$ is:

(C.06) $\nabla \cdot \mathbf{A} =$

$$+ \frac{\partial}{\partial w} (\alpha\beta A_w) \Big] \quad \text{(general orthogonal curvilinear coordinates)}$$

Substitution of the corresponding quantities from Figs. C.02 and C.03 shows that:

(C.07) $\nabla \cdot \mathbf{A} = \text{div} \, .$ $\qquad r \, \partial r \qquad \dfrac{1}{r}\dfrac{\partial A_\phi}{\partial \phi} + \dfrac{\partial A_z}{\partial z}$

$$\text{(cylindrical coordinates)}$$

(C.08) $\nabla \cdot \mathbf{A} = \text{div } \mathbf{A} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}$

$$\dfrac{1}{r \sin \theta}\dfrac{\partial A_\phi}{\partial \phi} \quad \text{(spherical coordinates)}$$

The Laplacian (∇^2) of a scalar V is formed by putting $\mathbf{A} = \text{grad } V$ in the formulas for **div A** and expanding. The results are:

$$\frac{\partial}{\partial w}\left(\frac{\alpha\beta}{\gamma}\frac{\partial V}{\partial w}\right)\Big] \quad \text{(general orthogonal curvilinear coordinates)}$$

$$\frac{\partial^2 V}{\partial z^2}$$

$$\text{(cylindrical coordinates)}$$

$$\overline{r^2 \sin^2 \theta \ \partial \phi^2} \quad \text{(spherical coordinates)}$$

From its definition, the curl of $\mathbf{A}$, $(\nabla \times \mathbf{A})$, can be calculated for the direction $\mathbf{u}_1$ by taking the line integral of $\mathbf{A}$ clockwise about the closed loop represented by dw, dv in Fig. C.04, and dividing by the area $\beta \ dv\gamma \ dw$. The line integral along a line through the point in this surface at which $\mathbf{A}$ is given, parallel to 1-2 is

The line integral along 1-2 is

$$4_v \ dv - \frac{\partial}{\partial w} \qquad\qquad dw$$

The line integral along the side 3-4 is

The line integral along the side 2-3 is

$$\left[\gamma A_w \ dw + \frac{\partial}{\partial v} \left(\gamma A_w \ dw \right) \frac{dv}{2} \right]$$

The line integral along the side 4-1 is

$$- \left[\gamma A_w \ dw - \frac{\partial}{\partial v} \left(\gamma A_w \ dw \right) \frac{dv}{2} \right]$$

The sum of these four contributions, divided by $\beta \ dv\gamma \ dw$, is the u component of $\nabla \times \mathbf{A}$. The v and w components can be written by cyclic rotation of $u \to v \to w$ and of $\alpha \to \beta \to \gamma$.

(C.12) $\nabla \times \mathbf{A} =$ $\qquad\qquad - \ \overline{} \ (\beta A_v) \mid \mathbf{u}_1$

$$1 + \overline{\alpha\beta}$$

(general orthogonal curvilinear coordinates)

The formulas corresponding to the systems shown in Figs. C.02 and C.03 are:

$$\text{(C.13)} \quad \nabla \times \mathbf{A} = \operatorname{curl} \mathbf{A} = \frac{1}{r}\left[\frac{\partial A_z}{\partial \phi} - \frac{\partial}{\partial z}\left(r A_\phi\right)\right]\mathbf{r_1}$$

$$+ \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\boldsymbol{\phi_1} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r A_\phi\right) - \frac{\partial A_r}{\partial \phi}\right]\mathbf{z_1}$$

$$\text{(cylindrical coordinates)}$$

$$\text{(C.14)} \quad \nabla \times \mathbf{A} = \operatorname{curl} \mathbf{A} = \frac{1}{r \sin \theta}\left[\frac{\partial}{\partial \theta}\left(A_\phi \sin \theta\right) - \frac{\partial A_\theta}{\partial \phi}\right]\mathbf{r_1}$$

$$+ \frac{1}{r \sin \theta}\left[\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}\left(r A_\phi \sin \theta\right)\right]\boldsymbol{\theta_1}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r A_\theta\right) - \frac{\partial A_r}{\partial \phi}\right]\boldsymbol{\phi_1} \quad \text{(spherical coordinates)}$$

Care must be taken in the derivation of formulas for three-dimensional problems that, because of symmetry, can be repre-
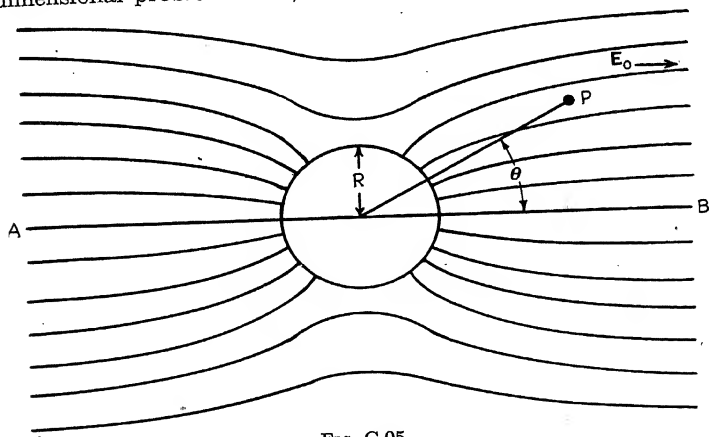
Fig. C.05.

sented on a plane. For example, if an uncharged conducting sphere is put into a region in which there was a uniform field E_0, the solution can be drawn on a plane because there is symmetry about a line $\overline{AB}$ drawn through the center of the sphere parallel to the applied field (see Fig. C.05). The potential V at any point such as P satisfies Laplace's equation. To obtain the proper function to equate to zero, note that if ϕ is the angle through which the figure must be rotated about the line AB to bring

any point, say P', into the plane of the paper, *V is independent of ϕ because the field is cylindrically symmetrical about AB.* Thus $\nabla^2 V$ for this problem is $\nabla^2 V$ for spherical coordinates, Eq. (C.11), with the last term omitted because $\partial V/\partial \phi = 0$. Expanding the remaining terms, the result is:

$$\frac{2}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2} + \quad ,- \cdot \frac{\partial V}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

for which a solution is, if $V = 0$ for $r = R$, and

$$-\frac{\partial V}{\partial r} = E_0 \cos \theta \qquad \text{for} \qquad r \to c$$

$$V = -$$

which can be checked by substituting.

APPENDIX D

ELECTROSTATIC POTENTIAL IN THE NEIGHBORHOOD OF VERY LONG ELECTRICALLY CHARGED CYLINDERS

D.01. Introduction.—The capacitance per unit length of two long equally charged cylinders is calculated in Sec. 11.04. It is the purpose of this appendix to analyze in more detail the electrostatic potential distribution in the neighborhood of such conductors.

In particular, the theoretical development that follows is planned specifically to demonstrate the important similarity between the calculation of capacitance from the concept of the electrostatic potential and the calculation of inductance from the concept of the vector potential; the latter is discussed in Appendix E.

D.02. Potential Due to a Charged Lineal Filament.—Consider a right cylinder of length 2λ and infinitesimal cross section dS in which there is a constant volume charge density ρ at every point. Such a cylinder will be referred to as a *lineal filament* of constant volume-charge density ρ. The total charge *in a unit length* of such a filament is

(D.01)   coulombs per meter

This charge will be referred to as the *lineal charge density* of this filament.

Designate by zz' the axis of this filament and designate as the xy plane the plane that is perpendicular to this axis at its middle point O; see Fig. D.01. Let P be any point in this xy plane at a distance N from O; this distance N will be referred to as the *normal distance* of P from the filament zz'. Let P_2 be a point on this filament and let z be the z coordinate of this point and let dz be an infinitesimal increment in z at this point. From Eq. (10.019) the electrostatic potential at P due to the entire charged filament of length 2λ is then

(D.02) $dV = \dfrac{1}{4\pi\epsilon_0} \displaystyle\int \dfrac{\rho \, dS \, dz}{\sqrt{z^2 + N^2}} = \dfrac{dQ}{2\pi\epsilon_0}$

This integral reduces to

(D.03) $dV = \dfrac{dQ}{2\pi\epsilon} [\log (\lambda + \sqrt{\lambda^2 + N^2}) - \log N]$

For $N << \lambda$, *i.e.*, for N very small compared with λ, one may write with negligible error

(D.04) $\sqrt{\lambda^2 + N^2} = \lambda$

and therefore, from the series expansion of $\log (1 + x)$ for $x^2 < 1$,

(D.05) $\log (\lambda + \sqrt{\lambda^2 + N^2}) = \log 2\lambda + \dfrac{1}{4}\left(\dfrac{N}{\lambda}\right)^2$

with a negligible error.
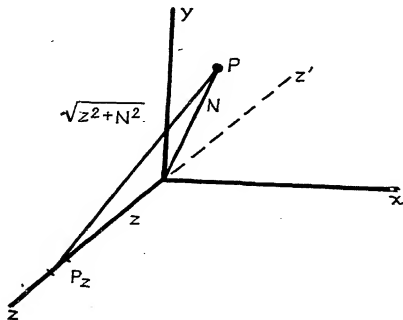
FIG. D.01.

Consequently Eq. (D.03) may be written with negligible error, when N is very small compared with the length of the filament,

D.03. Potential Due to a Charged Hollow Cylinder.—Consider next a right-circular cylindrical shell of length 2λ and internal radius R and thickness dR, and let this shell have a constant lume-charge density ρ at each point. Such a shell may be

considered as made up of contiguous filaments each of which has a lineal charge density

(D.07) $$dQ = \rho(R\,d\theta)\,dR$$

where θ and $d\theta$ are as indicated in Fig. D.02, the plane of which is perpendicular to the axis of this cylinder at its middle point O.
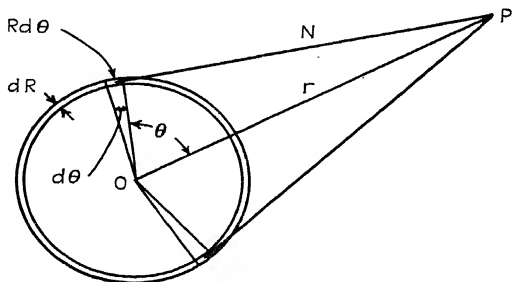
FIG. D.02.

Let r be the distance of any point P in this plane from the point O. From Eq. (D.06) the potential at P due to the entire charge between the inner and outer surfaces of this shell is

(D.08) $$dV = \frac{\rho R\,dR}{\pi\epsilon_0}\int_0^\pi \qquad\qquad d\theta$$

where

(D.09) $$\overline{\quad + R^2 - 2rR\,\cos\theta\quad}$$

Since the first term in the square brackets in Eq. (D.08) may be written

(D.10) $$\log\left(\frac{2\lambda}{N}\right) = \log(2\lambda) - \log N = \log 2\lambda - \frac{1}{2}\log N^2$$

it follows that

(D.11) $$\int_0^\pi \log\left(\frac{2\lambda}{N}\right)d\theta = \pi\log 2\lambda - \frac{1}{2}\int_0^\pi \log(r^2 + R^2 \\ - 2rR\cos\theta)\,d\theta$$

The integral in the right-hand member of Eq. (D.11) is of the form

$$\text{(D.12)} \qquad \int_0^\pi \log (a + b \cos \theta)\, d\theta$$

which has the value*

$$\text{(D.13)} \qquad \cdot \log \left(\frac{a}{\cdot} - \right. \qquad \text{for } a \gtrless b$$

in which $\sqrt{a^2 - b^2}$ is taken as the *positive* square root of $(a^2 - b^2)$. When for a and b are substituted the values

$$\text{(D.14)} \qquad \begin{aligned} a &= r^2 + R^2 \\ b &= -2rR \end{aligned}$$

the expression of Eq. (D.13) reduces to

$$\text{(D.15)} \qquad 2\pi \log r \qquad \text{for } r > R$$

and to

$$\text{for } r \gtrless R$$

Therefore Eq. (D.11) reduces to

$$\text{for } r > R$$

$$\text{for } r \gtrless R$$

Note next that

$$\text{(D.18)} \quad \int_0^\pi \frac{1}{4} \frac{\dot{N}^2}{\lambda^2}\, d\theta = \frac{1}{4\lambda^2} \int_0^\pi (r^2 + R^2 - 2rR \cos \theta)\, d\theta$$

Also note that the total charge per unit length in the cylindrical layer of internal radius R and thickness dR is

$$\text{(D.19)} \qquad d\bar{Q} = \rho 2\pi R\, dR$$

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* A method for calculating the integral is discussed in *Advanced Mathematics for Engineers;* H. W. Reddick and F. H. Miller; pp. 233–237, John Wiley & Sons, Inc., New York, 1938.

and therefore

(D.20) $$\rho R \, dr = \frac{d\bar{Q}}{2\pi}$$

Consequently Eq. (D.08) reduces to

(D.21) $$dV = \frac{d\bar{Q}}{2\pi\epsilon_0}\left(\frac{2\lambda}{r} + \frac{\prime}{} \quad \frac{R^2}{4\lambda^2}\right) \qquad \text{for } r > R$$

$$dV = \frac{d\bar{Q}}{} \qquad \frac{2\lambda}{} \qquad \frac{}{4\lambda^2} \qquad \text{for } r \lessgtr R$$

When both the radius R of the shell and the normal distance r of the point P from the axis of the shell are so small that the term $r^2 + R^2/4\lambda^2$ may be neglected in comparison $\log 2\lambda/r$ for all values of r equal to or greater than R, then Eq. (D.21) may be written

(D.22)   for $r \geq R$

and

(D.23) $$dV = \frac{d\bar{Q}}{} \quad \left(\frac{2\lambda}{R}\right) \qquad \text{for } r$$

A comparison of Eq. (D.22) with Eq. (D.06) shows that under the conditions stated the potential produced at any point *outside* the charged cylindrical shell is identical with that which would be produced were all the charge between the inside and outside surfaces of this shell concentrated along its *axis*. This relation, however, is not true for points *inside* the space enclosed by the shell, since from Eq. (D.23) the potential at every point within this space has the *same* value, a value that is identical with the potential at the surface of this shell. Also note that, if there are no other charges that affect the potential of the shell, this shell is an equipotential surface and the other equipotential surfaces in the field are cylindrical surfaces concentric with the given shell.

When the cylindrical shell under consideration is the contact layer between a conductor and a dielectric, the contact layer having a thickness that is a "macroscopic" infinitesimal ΔR, and the *charge per unit length* of this shell is a noninfinitesimal number Q, then the potential at the point P is likewise a noninfinitesimal number V which, for all points *outside* the

surface of this shell, is

$$(D.24) \qquad V_o = \frac{Q}{2\pi\epsilon_o} \log\left(\frac{-\cdots}{r}\right) \qquad \text{for } r \gtrless (R + \Delta R)$$

and for all points *inside* the inner surface of this shell is

$$(D.25)$$

By an extension of the above analysis it may be shown that Eqs. (D.24) and (D.25) are applicable to points in any plane that is perpendicular to the axis of the cylindrical shell and that intersects this axis at a point that lies between the two ends of the cylinder at a distance from each end very large in comparison with both the radius R of the shell and the normal distance r from the point P to its axis.

D.04. Potential Due to a Solid Cylindrical Charge.—Consider next a solid cylinder of length 2λ and radius R_0, and let there be a constant volume-charge density ρ at each point throughout this cylindrical volume. Such a volume may be considered as made up of contiguous coaxial cylindrical shells. Let R be the internal radius and dR the thickness of any one of these shells. Then the total charge in this cylindrical layer is

$$(D.26) \qquad dQ = \rho 2\pi R \, dR$$

From Eq. (D.22) the potential at any point P at a normal distance r from the axis of this cylinder, *when r is greater than* R_0, is

$$(D.27) \quad V_0 = \left[\frac{1}{2\pi\epsilon_0} \log\left(\frac{2\lambda}{r}\right)\right] \int_0^R dQ = \frac{\rho\pi R^2}{2\pi\epsilon_0} \log\left(\frac{2\lambda}{r}\right)$$

But $\rho\pi R^2$ is the *total charge Q per unit length* of this cylinder. Therefore,

$$(D.28) \qquad V_0 = \frac{Q}{2\pi} \log\left(\frac{2\lambda}{r}\right. \qquad \text{for } r >$$

In a similar manner, when P is inside the cylinder of radius R_0 at a distance R_i from its axis, then the potential at this point due to that portion $(R_i^2/R_0^2)Q$ of the total charge per unit length inside a cylinder which has a radius R_i is

$$(D.29) \qquad V_i' = \frac{QR_i^2}{2\pi\epsilon_0 R_0^2} \log\left(\frac{2\lambda}{R_i}\right)$$

From Eq. (D.23) the potential at this point due to the portion $(2R\,dR/R_0^2)Q$ of the total charge in a shell of internal radius R and thickness dR, where $R_i \lessgtr R \lessgtr R_0$, is

(D.30) $$dV_i'' \qquad 2Q \quad R \log\left(\frac{2\lambda}{R}\right) dR$$

The potential at a normal distance R_i from the axis of the cylinder, due to that portion of the total charge per unit length which is in the shell of internal radius R_i and external radius R_0, is then the definite integral of Eq. (D.28) from $R = R_i$ to $R = R_0$. Since

(D.31) $$\int R \log R\, dR = \tfrac{1}{2}R^2 \log R - \tfrac{1}{4}R^2$$

the definite integral is

(D.32) $$V_i'' = \frac{Q}{2\pi\epsilon_0}\left[\log\left(\frac{2\lambda}{R_0}\right) - \frac{R_i^2}{R_0^2}\log\left(\frac{2\lambda}{R_i}\right) + \frac{R_0^2 - R_i^2}{2R_0^2}\right]$$

The total potential at the point which is inside the cylinder of radius R_0 at a distance R from its axis is then the sum of Eqs. (D.29) and (D.32), *viz.*,

(D.33) $$V_i = \frac{Q}{2\pi\epsilon_0}\left[\log\left(\frac{2\lambda}{R_0}\right) + \frac{R_0^2 - R_i^2}{2R_0}\right]$$

The *average value* of this internal electrostatic potential over the entire cross section of the cylinder of radius R_0 within which the volume-charge density has the constant value $\rho = Q/\pi R_0^2$ at every point is $Q/2\pi\epsilon_0$ times the average value of $(R_0^2 - R_i^2)/2R_0$ relative to the area of the cross section of this cylinder, which latter is

(D.34) $$\frac{1}{\pi R_0^2}\int_0^{R_0}\left(\frac{R_0^2 - R_i^2}{2R_0^2}\right) 2\pi R_i\, dR_i = \frac{1}{4}$$

Therefore the average value of V_i in a right-circular cylinder that has a uniform volume-charge density $\rho = Q/\pi R_0^2$ is

(D.35) $$V_{ia} = \frac{Q}{2\pi\epsilon_0}\left(\log\frac{2\lambda}{R_0} + \frac{1}{4}\right)$$

The range in the average of V_i is from

(D.36) $$V_{1a} = \frac{Q}{} \log \left(\frac{2\lambda}{R} \right.$$

at the axis of the cylinder to

(D.37) $$V_{1s} = \frac{Q}{2\pi\epsilon_0} \log \left(\frac{2\lambda}{R_0} + \frac{1}{2} \right)$$

at the surface of this cylinder.

D.05. Potential Due to Two Parallel Cylindrical Conductors and the Capacitance of the Capacitor Formed by Them.—Consider next two parallel cylinders whose radii are respectively R_1 and R_2, and let P be a point outside of each cylinder at a distance r_1 from cylinder No. 1 and at a distance r_2 from cylinder No. 2. Let these two cylinders be charged respectively to Q coulombs per meter and to $-Q$ coulombs per meter.   Then if these charges
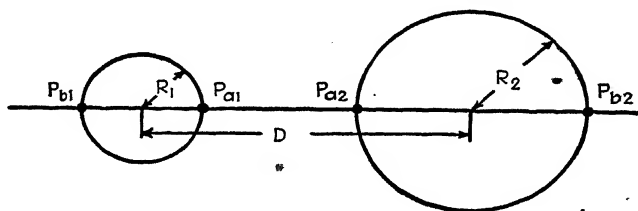
Fig. D.03.

are confined to the outer surface layers of the two cylinders and have a uniform surface density, the electrostatic potential at the point P is, from Eq. (D.24),

(D.38) $$\frac{Q}{2\pi\epsilon_0}$$

From this relation it follows that the electrostatic potential at P_{a1} in Fig. D.03 is

(D.39) $$V_{a1} = \frac{Q}{2\pi\epsilon_0} \log \frac{D - R_1}{R_1}$$

and the potential at P_{a2} is

(D.40) $$V_{a2} = \frac{Q}{2\pi\epsilon_0} \log \frac{R_2}{D - R_2}$$

The difference of potential between P_{a1} and P_{a2} is therefore

(D.41) $V_{a12} = \frac{Q}{2\pi\epsilon}\, \log$

In a similar manner the difference of potential between P_{b1} and P_{b2} is

These two potential differences V_{a12} and V_{b12} are obviously not the same, and therefore neither the surface of cylinder No. 1 nor that of cylinder No. 2 is an equipotential surface. However, if the two cylinders are two insulated conductors charged to Q coulombs and $-Q$ coulombs respectively they must be equipotential surfaces if the charges are at rest. This means that under static conditions the charges on the two cylinders cannot have a uniform surface distribution, although this condition is approximated when R_1 and R_2 are both so small that $\log (D - R_1)$ and $\log (D + R_1)$ may each be taken with negligible error equal to $\log D$ and $\log (D - R_2)$ and $\log (D + R_2)$ may each be taken with negligible error equal to $\log D$. Under these conditions the difference of potential between any point on cylinder No. 1 and any point on cylinder No. 2 is, with negligible error

(D.43) $V_{12} = \dfrac{Q}{\quad} \log \dfrac{}{R_1 R_2}$

In particular, when $\ =R_2 = R$ this reduces to

(D.44) $V_{12} = \dfrac{Q}{\pi\epsilon_0} \log \dfrac{\nu}{n}$

Therefore the capacitance per meter between two equal parallel cylinders each of radius R meters, whose axes are a distance D meters apart, and which are separated by a dielectric of permittivity ϵ_0 throughout, is

(D.45) $\dfrac{}{\log \dfrac{D}{R}}$ farads per meter

When the dielectric between the two cylinders is homogeneous and isotropic and has any other permittivity ϵ, then the capaci-

tance is likewise given by Eq. (D.45), provided that ϵ is substituted for ϵ_0.

The exact formula for the capacitance per unit length of two long parallel conducting cylinders may be derived as follows. From Eq. (D.06) the potential at any point P *outside* each of two parallel lineal filaments, when charged to Q and $-Q$ coulombs per unit length respectively is

$$\text{(D.46)} \qquad V =$$

where r_1 and r_2 are respectively the normal distances of P from these filaments. Designate by D_0 the distance apart of these filaments, and choose as the origin in the XY plane the point at which filament No. 1 intersects this plane. Then, in terms of the coordinates x and y of the point P,

$$\text{(D.47)} \qquad \begin{aligned} r_1 &= \sqrt{x^2 + y^2} \\ r_2 &= \sqrt{(D_0 - x)^2 + y^2} \end{aligned}$$

and Eq. (D.46) may be written

$$\text{(D.48)} \qquad V = \frac{Q}{4\pi\epsilon_0} \log H$$

where

$$\text{(D.49)} \qquad H = \frac{(D_0 - x)^2 + y^2}{x^2 + y^2}$$

Note that for all pairs of values of x and y for which H has a constant value the electric potential V is likewise a constant. Therefore all those points for which H is a constant form an equipotential surface. The family of curves whose equations are given by Eq. (D.49) when to H are assigned successive constant values are the intersection of these equipotential surfaces with the xy plane. This equation, which may also be written

$$\text{(D.50)} \qquad x^2 + y^2 = \frac{D_0^2 - 2xD_0}{H - 1}$$

is the equation of a circle whose center is on the x axis at a point whose abscissa is

$$\text{(D.51)} \qquad \alpha = \frac{D_0}{1 - H}$$

and whose radius R is the positive square root of

$$(D.52) \qquad R^2 = \frac{D_0^2 H}{(1 - H)^2}$$

Therefore the equipotential surfaces outside the two charged cylindrical surfaces are right circular cylinders which surround one of the two lineal filaments and whose intersections with the xy plane are all circles with their centers on the x axis, each circle having a different center.

The abscissa α of the center of any one of these circles is readily found by eliminating H from Eqs. (D.51) and (D.52), which gives

$$(D.53) \qquad \alpha = \frac{D_0 \pm \sqrt{D_0^2 + 4R^2}}{2}$$

There are therefore two equipotential surfaces of a given radius R, for one of which α has the *negative* value

$$(D.54) \qquad \alpha_1 = - \frac{}{} \qquad \left. \frac{2R}{D_0}\right)^2 - 1 \right]$$

and which surrounds filament No. 1, and for the other α has the positive value

which equipotential surface surrounds filament No. 2. The center of the first circle is at a distance (α_1) to the left of filament No. 1 and the second is at the same distance to the right of filament No. 2. The normal distance between the axes of these two equipotential surfaces is

$$(D.56) \qquad D = \alpha_2 - \alpha_1 \qquad \sqrt{D_0^2 + 4R^2}$$

which solved for D_0 gives

$$(D.57) \qquad D_0 = \sqrt{D^2 - 4R^2}$$

Substitute this expression for D_0 in Eqs. (D.54) and (D.55) and put

$$(D.58) \qquad W = \frac{2R}{D}$$

There results

(D.59) $$\alpha_1 = \frac{D}{2}[\sqrt{1 - W^2} - 1]$$

(D.60) $$\alpha_2 = \frac{D}{2}[\sqrt{1 - W^2} + 1]$$

The abscissa of the point to the right of the first filament at which the circle of radius R around this filament intersects the x axis is then

Therefore the value of H for this circle is, from Eq. (D.49),

$$\overline{x_1} \quad , \quad \left[\frac{\sqrt{1 + W} + \sqrt{1 - W}}{\sqrt{1 + W} - \sqrt{1 - W}}\right]^2$$

Similarly, the abscissa of the point to the left of the second filament at which the circle of radius R about that filament intersects the x axis is

(D.63) $$\imath_2 = \alpha_2 - R = \frac{D}{2}[\sqrt{1 - W^2} + (1 - W)]$$

and for this circle the value of H is

(D.64) $$x_2 \qquad \left(\frac{\sqrt{1 + W} - \sqrt{1 - W}}{\sqrt{1 + W} + \sqrt{1 - W}}\right)^2.$$

From Eq. (D.48) the *difference* of potential between the two cylinders, each of radius R, one surrounding the first filament and the other surrounding the second filament, is then

(D.65) $$V = \frac{Q}{\pi\epsilon_0} \log\left(\frac{\sqrt{1 + W} + \sqrt{1 - W}}{\sqrt{1 + W} - \sqrt{1 - W}}\right)$$
$$= \frac{Q}{\pi\epsilon_0} \log\left[\frac{D}{2R} + \sqrt{\left(\frac{D}{2R}\right)^2 - 1}\right]$$

The last expression may be written

(D.66) $$V_{12} = \frac{Q}{\pi\epsilon_0} \cosh^-$$

From the principle that, if the potential due to any distribution
of electric charges on the bounding surface (or surfaces) of a
given region is known, then the electrostatic field at every point
in this region is uniquely determined irrespective of how the
potentials at the several parts of the bounding surface (or sur-
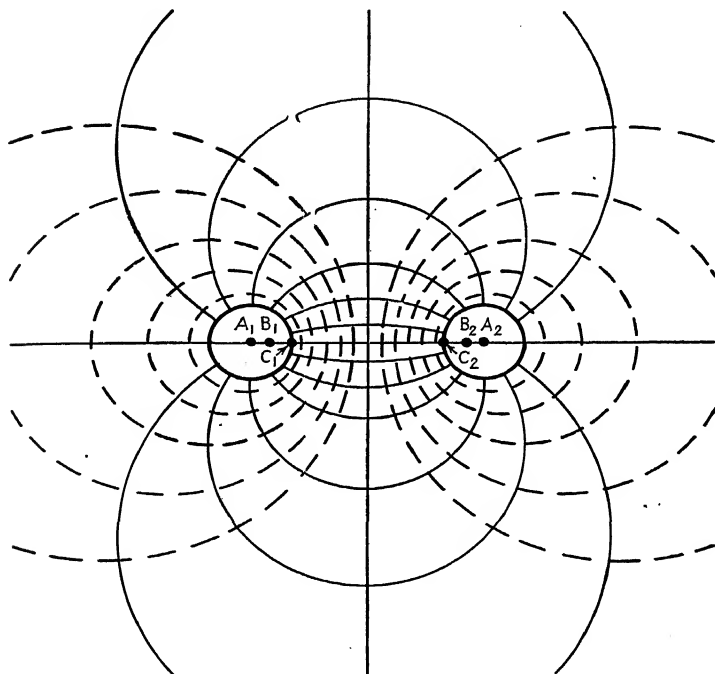
FIG. D.04.

faces) are established, it follows that, if two cylinders of equal
radius R, whose axes are at a distance D apart, are given charges
Q and $-Q$ coulombs per unit length, then V is the difference of
potential in volts between these surfaces. Consequently the
capacitance per unit length of a condenser formed by two circular
cylinders each of radius R meters, when their axes are D meters
apart and the permittivity of the medium which surrounds them
is ϵ_0, is

(D.67)
$$C = \frac{\pi \epsilon_0}{\cosh^{-1}\left(\frac{\cdot}{\cdot\cdot}\right)}$$
          farads per meter

When the surrounding dielectric is *homogeneous and isotropic* and has a permittivity ϵ, then for ϵ_0 in Eq. (D.67) must be substituted this permittivity ϵ.

By an extension of the above analysis one can find the surface density of the electric charge at each point on the surface of each cylinder.* This analysis shows that the charge density is a maximum at the points on the two cylinders that are the closest together and a minimum at the points that are the farthest apart.

The lines of the electric-field intensity and the lines representing equipotential surfaces for the problem of two long parallel conductors, which are cylinders having equal diameters, are shown schematically in Fig. D.04. The solid lines in this figure represent lines of the electric-field intensity, except the heavy solid lines, which represent the intersections of the surfaces of the two conductors with the plane of the paper. The dashed lines are traces of equipotential surfaces with the plane of the paper. Points A_1, A_2 are centers of the conductor cylinders; points B_1, B_2 represent intersections of the lines along which hypothetical charges would be uniformly distributed to make the conductor surfaces equipotential surfaces. Integration of **E** from C_1 to C_2 leads conveniently to the difference of potential from which Eq. (D.67) can be obtained.

* See *The Capacity between Two Equal Parallel Wires;* Harold Pender and H. S. Osborne; *Elec. World,* **56,** 667–670, Sept. 22, 1910.

THE VECTOR POTENTIAL AS THE BASIC QUANTITY FOR THE CORRELATION OF MAGNETIC PHENOMENA

E.01. Introduction.—It was pointed out in Sec. 12.01 that electromagnetic phenomena can be concisely correlated in terms of two functions—a scalar potential V and a vector potential $\mathbf{A}$. The scalar potential V, which is the same as the electrostatic potential when the electric field is not varying with time, has been discussed in detail in the preceding sections of this book. It is the purpose of this appendix to show how the usual magnetic concepts can be derived from the basic concept of vector potential and how the phenomena due to *varying* electric and magnetic fields can be correlated in terms of this potential and the electric scalar potential. The concept of *magnetic* scalar potential is also introduced.

E.02. Vector Potential, Magnetic Flux, and Magnetically Induced Electromotive Force.—In Chaps. X and XI and in Appendix D it was shown that those phenomena known as electrostatic phenomena may be coordinated mathematically in terms of the volume integral of the ratio of charge density to distance, *viz.*, in terms of the *scalar* function

(E.001)
$$V = \frac{}{4\pi\epsilon_0} \qquad \text{volts}$$

In a similar manner magnetic phenomena may be coordinated mathematically in terms of a function that is also a volume integral, but in this case the *integral is itself a vector*. To distinguish between the two volume integrals by short names, the function V defined by Eq. (E.001) is called the *scalar potential* owing to the magnitudes and positions of the electric charges in the region under consideration, and the vector integral that will now be defined is called the *vector potential* attributable to the magnitudes and velocities of the electric charges in this region.

When the velocities of the charges in a given volume v_a do not vary too rapidly with time, the vector potential may be defined as follows. Let ρ be the electric-charge density in a moving charge that at any instant of time occupies a fixed infinitesimal volume dv_a but that is moving with a velocity **u**. Designate by u_x, u_y, and u_z the components of this velocity in the positive senses of the three axes in a right-handed rectangular cartesian system of coordinates. Designate by μ_0 a constant appropriately chosen to suit the system of units in which the charge, velocity, etc., are expressed, but which is independent of the nature of the medium in which the charges are located. In the rationalized mks system of units, μ_0 is assigned the value

(E.002) $$\mu_0 = 1.257 \times 10^{-6}$$ henry per meter

This constant is called the *magnetic permeability of free space.*

Designate by P any point fixed relative to dv_a and designate by r the distance of this point from dv_a. Designate by v_a the total volume inside a fixed boundary. Then the vector

(E.003) $$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z$$

of which the three axial components are

(E.004)

$$A = \frac{\mu_0}{4\pi} \int_{v_a} \frac{u_{xi}}{r}$$

$$A_y = \frac{\mu_0}{4\pi} \quad \frac{u_y \rho_a \, \ell}{}$$

$$A \quad - \quad \frac{\mu_0}{} \quad \frac{u_z \rho_a}{}$$

webers per meter

is defined as the *vector potential* at the point P owing to all the moving charges in the specified volume v_a. The necessary modification in these expressions to take account of rapidly varying velocities is given in Sec. E.04.

A briefer notation for the vector **A** is

(E.005) $$\mathbf{A} = \frac{\mu_0}{4}$$ webers per meter

where the integral sign is interpreted as indicating the *vector sum* of the vectors $\left(\dfrac{\rho_a \, dv_a}{r} \right)$ **u** for all the infinitesimal volumes

dv_a which lie within the fixed volume v_a. In this expression $\mathbf{u}$ is the vector

(E.006) $$\mathbf{u} = \mathbf{i}u_x + \mathbf{j}u_y + \mathbf{k}u_z$$

whose magnitude is the magnitude of the linear velocity of dv_a and whose direction is the direction of this velocity.

Designate by ϕ_c the value of the *circulation* of the vector potential $\mathbf{A}$ around any closed loop C, *viz.*,

(E.007) $\mathbf{ds}$ webers

This scalar ϕ_c is the quantity in terms of which may be expressed the measurable magnetic effects attributable not only to electric currents, but also to permanent magnets, electromagnets, and moving electric charges in general.   Although as here defined ϕ_c is the circulation of a vector $\mathbf{A}$ around a closed loop C, it may also be expressed in terms of the flux, through any surface for which C is the total boundary, of another vector $\mathbf{B}$ derivable from $\mathbf{A}$ in a relatively simple manner.   For this reason the quantity ϕ_c is called the *magnetic flux* through the closed loop C.

The importance of the concept of the vector potential $\mathbf{A}$ lies in the fact that in terms of the rate of change of this vector with respect to time may be expressed the force that is exerted upon an electric charge at rest owing to the *magnitudes* and *velocities* of other charges in its vicinity.   The importance of the concept of magnetic flux, ϕ_c, lies in the fact that in terms of the rate of change of this scalar may be expressed the emf that is induced in any closed path owing to the *magnitudes* and *velocities* of the electric charges both in this path and in the vicinity of this path.

In Chap. X the force per unit positive charge exerted upon an infinitesimal charge dQ at a point P due to the *magnitudes* and *positions* of other charges in its vicinity was designated by the symbol $\mathbf{E}$, and to this force per unit positive charge the name *electrostatic* intensity was given.   In this appendix the symbol $\mathbf{E}'$ will be used for this electrostatic intensity and the symbol $\mathbf{E}''$ will be used for the force per unit positive charge that would be exerted upon a charge dQ *held at rest* at a point P, owing to the *magnitudes* and *velocities* of the other charges in its vicinity. This intensity $\mathbf{E}''$ will be called the *electromotional intensity* at P, in contradistinction to the electrostatic intensity $\mathbf{E}'$.   The vector

sum of these two intensities at any point, *viz.*, the vector

(E.008) $\mathbf{E} = \mathbf{E}' + \mathbf{E}''$ volts per meter

is called the *electric intensity* at this point, and wherever this vector $\mathbf{E}$ has a magnitude different from zero there is said to exist an *electric field.* *

Experiment justifies the assumption that the electromotional intensity at any point P is a vector that has a magnitude equal to the rate of change, with respect to time, of the vector potential $\mathbf{A}$ at this point, and that has the *same* sense as this vector $\mathbf{A}$ when $\mathbf{A}$ is *decreasing* with time and the opposite sense when $\mathbf{A}$ is increasing with time, *viz.*,

(E.0.09)

The circulation of this electromotional intensity around a closed loop C is equal to the time rate of decrease of the circulation of the vector $\mathbf{A}$ around this same loop, *viz.*,

(E.010) $'' \cdot \mathbf{ds} = - \frac{\sigma}{\sim}$ $\mathbf{ds} = - \frac{d\phi_c}{dt}$

This scalar

(E.011) $\frac{d\phi_c}{dt}$ volts

is the work per unit charge that is (or would be) done by the electromotional intensity $\mathbf{E}''$ in moving an infinitesimal positive charge completely around the closed loop C. The scalar e_c defined by (E.010) is called the *magnetically induced electromotive force* in the specified loop C. This relation between magnetic flux and magnetically induced emf was first stated by Faraday, who, however, defined magnetic flux in a much more devious (but nevertheless equivalent) manner. For this reason the relation expressed by Eq. (E.011) is known as *Faraday's law of electromagnetic induction*. It is of fundamental importance in practically all electrical engineering problems.

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* When for V and $\mathbf{A}$ are used the modified formulas that take into account rapidly changing velocities, then $\mathbf{E}'$ and $\mathbf{E}''$ must be taken respectively the gradient and time rate of charge (in the proper sense) of these modified potentials (see Sec. E.04).

E.03. Magnetic-flux Density B, the Curl of a Vector and the Cross Product of Two Vectors.—The proof that the circulation ϕ_c of the vector potential **A** around any closed loop C is equal to the flux of a vector **B** through any surface for which this loop is the total boundary will now be given. Consider first *any* vector

(E.012) $$\mathbf{A} = \mathbf{i} A_x$$

in which the three scalars A_x, A_y, A_z are each finite and single-valued with respect to the three space coordinates x, y, and z in a right-handed rectangular cartesian system, and whose first-order partial derivatives with respect to these coordinates are each also finite and single-valued. It can be shown that the
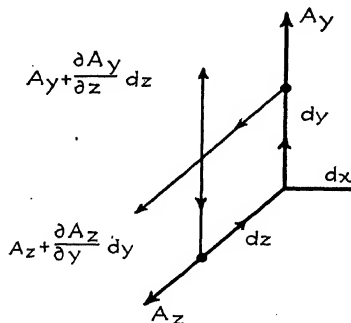
FIG. E.01.

scalars A_x, A_y, and A_z defined by Eq. (E.004) are such functions, provided the charge density ρ_a and velocity **u** are each finite and continuous at every point in the volume v_a. Designate by P any point whose coordinates are x, y, and z, and draw from P a line of length dx parallel to and in the positive sense of the x axis, and imagine a rectangular area $dy\,dz$ at P in a plane which is parallel to the yz plane and therefore perpendicular to dx. Consider a closed loop C formed by the sides of this rectangle, traced in the sense of the arrows on these sides in Fig. E.01. The sense in which this loop is traced then bears a right-handed-screw relation to the normal drawn from this area in the positive sense of the x axis. The circulation of the vector **A** around this loop C, in the right-handed-screw sense relative to the normal

drawn in the positive sense of the x axis, is then

$$(E.013) \quad A_y \, dy + \left(A_z + \frac{\partial A_z}{\partial y} \qquad dz \right.$$

This sum reduces to

$$(E.014) \qquad\qquad \frac{\partial}{\partial y} - \frac{\partial A_y}{\partial z} \Bigg) dy \, dz$$

which may also be written

$$(E.015) \qquad\qquad d\phi_x = B_x \, dS$$

where dS is the elementary area $dy \, dz$ and

$$(E.016) \qquad\qquad B_x \qquad \overline{\frac{}{\partial y}} \qquad \partial z$$

Since any area in a surface of any shape may be considered as made up of contiguous rectangles each of which has sides of infinitesimal length, and since the relation in Eq. (E.004) is independent of the choice of the coordinate axes, the circulation of the vector **A** around the boundary of any plane area dS may be written

$$(E.017) \qquad\qquad d\phi_n = B_n \, dS \qquad .$$

where B_n is the component of the vector

$$(E.018) \qquad\qquad \mathbf{B} = \mathbf{i}B_x + \mathbf{j}B_y + \mathbf{k}B_z$$

which is normal to the surface dS in the right-handed-screw sense relative to the sense in which this circulation is taken, and B_x, B_y, and B_z are scalar functions that at the point P have the values

$$(E.019) \qquad\qquad B_z = \begin{array}{c} \dfrac{\partial A_z}{\partial y} \\[6pt] \dfrac{\partial A_x}{\partial z} \qquad \overline{\partial x} \\[6pt] \dfrac{\partial A_y}{\partial x} \qquad \dfrac{\partial A_x}{\partial y} \end{array}$$

From the definition of circulation, if two surfaces such as dS and dS' have a common boundary, that *part* of the circulation of a vector **A** around dS which corresponds to this common boundary has the same magnitude but the opposite algebraic sign to that part of the circulation of **A** around dS', provided in each case the circulation is taken in the same sense around dS and dS' when viewed from a point that is on a given side of the total surface formed by dS and dS'. Hence the circulation in this same sense around the boundary of the total surface formed by dS and dS' is the sum of the circulations around the (*outside*) boundary of the two individual surfaces. Any closed loop C may be considered as the boundary of the total surface made up of contiguous infinitesimal plane area dS. Therefore the total circulation of **A** around any closed loop C is

$$(E.020) \qquad \phi_c = \oint_C \mathbf{A} \cdot d\mathbf{s} = \int_S B_n \, dS = \int_S \mathbf{B} \cdot d\mathbf{S}$$

Therefore ϕ_c is the total flux of the vector **B**, whose components are defined by Eq. (E.019), over any surface whatever whose complete boundary is the closed loop C.

In the development given above, suppose **A** is the vector potential, **B** is the magnetic-flux density, and ϕ_c is the flux through the surface S bounded by the closed loop C. Wherever the magnetic-flux density **B** has a value different from zero there is said to exist a *magnetic field*.

When the components of a vector **B** bear to another vector **A** the relations given by Eq. (E.019) then **B** is said to be the curl of the vector **A**. This relation is usually expressed in vector notation by the equality

$$(E.021) \qquad\qquad \mathbf{B} = \nabla \times \mathbf{A}$$

where ∇ is the vector operator defined by Eq. (10.027) of Sec. 10.04, and the cross between ∇ and **A** signifies the *cross product* of the two quantities,

$$(E.022) \qquad\qquad \nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

$$(E.023) \qquad\qquad \mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z$$

in the sequence indicated. In performing such a *cross multiplication*, each of the quantities defined by Eqs. (E.022) and (E.023) is considered as an algebraic polynomial and to the

cross products of the unit vectors **i**, **j**, and **k** are assigned the values

(E.024)
$$\begin{array}{lll}
\mathbf{i} \times \mathbf{i} = 0 & \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{i} = -\mathbf{k} \\
\mathbf{j} \times \mathbf{j} = 0 & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} \\
\mathbf{k} \times \mathbf{j} = 0 & \mathbf{k} \times \mathbf{i} = & \mathbf{i} \times \mathbf{k} = -\mathbf{j}
\end{array}$$

When these conventions are adopted, the cross product $\mathbf{\nabla} \times \mathbf{A}$ results in a vector whose components in the positive senses of the x, y, and z axes are as given by Eq. (E.019). In other words, the notation $\mathbf{B} = \mathbf{\nabla} \times \mathbf{A}$ means nothing more than that $\mathbf{B}$ is a vector whose axial components are given by Eq. (E.019).

The concept of cross product is also frequently useful in other relationships between two vectors $\mathbf{A}$ and $\mathbf{B}$, irrespective of what physical quantities are represented by these two vectors; *i.e.*, if $\mathbf{A}$ and $\mathbf{B}$ are any two vectors

(E.025) $$\mathbf{A} = \mathbf{i}A_x$$
(E.026) $$\mathbf{B} = \mathbf{i}B_x$$

then the cross product of $\mathbf{A}$ by $\mathbf{B}$ with $\mathbf{A}$ as the prefactor is defined as the vector

(E.027)

where

(E.028)
$$\begin{aligned}
C_x &= A_y B_z - A_z B_y \\
C_y &= A_z B_x - A_x B_z \\
C_z &= A_x B_y - A_y B_x
\end{aligned}$$

Although these expressions for the components of a cross product may at first sight seem confusing with respect to subscripts, one may avoid errors by noting that the sequence of the subscripts to C and to the first terms in the right-hand members follow the sequence *xyzxyz*, and that the subscripts to the second terms in the right-hand members are the reverse of those of the first terms. Also note that the cross product of $\mathbf{A}$ by $\mathbf{B}$ with $\mathbf{B}$ as the *prefactor* is the negative of $\mathbf{A} \times \mathbf{B}$, *viz.*,

(E.029) $$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}$$

The relation expressed by Eq. (D.020) when $\mathbf{A}$ and $\mathbf{B}$ are any two vectors may be written

(E.030) $$\times \mathbf{A}) \cdot \mathbf{dS}$$

i.e., the circulation of any vector around a closed loop C is always equal to the flux of the *curl* of this vector through any surface whose complete boundary is this loop C. This relation is known as *Stokes's theorem*. Compare with Gauss's theorem, *viz.*,

(E.031)

where S is the bounding surface of the volume v and $\nabla \cdot \mathbf{A}$ is the *divergence* of $\mathbf{A}$.

E.04. Retarded Scalar and Vector Potentials.—As pointed out in Sec. E.02, when the moving charges in a specified volume v_a have a high velocity, then, to coordinate mathematically the observed phenomena, it is necessary to modify Eq. (E.001) for the scalar potential and Eq. (E.005) for the vector potential. In form, the necessary modifications are very simple, *viz.*, that in the expression

(E.032) $V = \qquad \rho_a]$ volts

for the scalar potential at a point P at any time t, one must use for $[\rho_a]$ the value of the charge density in the elementary volume dv_a the charge density *which was at dv_a at a time r/c earlier*, where c is the velocity of light in free space. The square brackets are the symbols of this specification. Similarly, in the expression

(E.033) $[\rho_a \mathbf{u}]$ webers per meter

for the vector potential at a point P at time t one must use for ρ_a the charge density *which was at dv_a at a time r/c earlier* and for $\mathbf{u}$ the velocity at which the charge in dv_a was moving *at this earlier instant of time*. The square brackets are the symbols of this specification. When these modifications have been made the two potentials V and $\mathbf{A}$ *both* become functions of time as well as of the space coordinates x, y, and z. These modified potentials are usually referred to as *retarded potentials*.

The observed phenomena in electric and magnetic fields require that, in terms of the retarded scalar potential V at a given point P and the retarded vector potential $\mathbf{A}$ at this point, the total electric intensity at this point P must be considered as the vector

sum of the intensities

(E.034) volts per meter

(E.035) $$\mathbf{E}'' = \frac{\partial \mathbf{A}}{\partial t}$$ volts per meter

Any region of space in which these two intensities exist simultaneously is called an *electromagnetic field*.

The relations expressed by Eqs. (E.034) and (E.035) are identical with those already developed for the case when there is no rapid variation in the velocities of the electric charges in the field. When Eqs. (E.034) and (E.035) are combined there results for the total electric intensity at the point P the expression

(E.036) $$\mathbf{E} = -\boldsymbol{\nabla}V - \frac{\partial \mathbf{A}}{\partial t}$$ volts per meter

This is one of the basic equations for all electromagnetic theory (see Sec. 15.05). Also note that when it is necessary to employ the retarded vector potential the symbols $\mathbf{E}$ and $\mathbf{B}$ must be taken as representing the electric intensity and magnetic-flux density as derived from these retarded potentials.

From the relation

(E.037) $$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$ webers per square meter

between the magnetic-flux density $\mathbf{B}$ and the vector potential $\mathbf{A}$, and the fact that the electromotional intensity $\mathbf{E}''$ is

(E.038) $$\frac{}{\partial t}$$ volts per meter

it follows that

(E.039) $$\boldsymbol{\nabla} \times \mathbf{E}'' = \frac{\partial \mathbf{B}}{\partial t}$$ volts per square meter ·

Also, since the curl of a gradient is zero, it follows that the curl of the electropositional intensity $\mathbf{E}'$ is zero, *viz.*,

(E.040) $$\boldsymbol{\nabla} \times \mathbf{E}' = 0$$

Therefore the curl of the total electric intensity $\mathbf{E} = \mathbf{E}' + \mathbf{E}''$ is equal to minus the time rate of increase of the magnetic-flux density $\mathbf{B}$, *viz.*,

(E.041) volts per square meter

E.05. Magnetic Polarization and Electric Displacement Current.—It can be shown that, consistent with the concept of retarded potentials, the circulation of the magnetic-flux density B around any closed curve C may always be expressed as the product of μ_0 by the integral of the vector

(E.042) $$\rho\mathbf{u} + \frac{\partial}{\partial t}(\epsilon_0\mathbf{E}) \quad \text{amperes per square meter}$$

over *any* surface bounded by this curve C, *viz.*,

(E.043) amperes

In both Eqs. (E.042) and (E.043) the symbol ρ stands for the volume density of the electric charge which at any instant is moving through a point in the surface, u is the velocity of this charge, ϵ_0 and μ_0 are respectively the permittivity and permeability of free space, and E and B are respectively the electric intensity and magnetic-flux density derived from the retarded potentials as just explained. In particular note that ρ is the charge density, irrespective of whether the charge in question is a free charge or a bound charge, and u is the linear velocity of a point in this charge, irrespective of whether this charge as a whole is moving from molecule to molecule or is undergoing a varying linear displacement within a molecule or has a rotary motion within this molecule.

In order to coordinate the observed phenomena in a material medium it is convenient to consider separately these three kinds of motion of electric charges. Let ρ_f and $\mathbf{u}_f$ be respectively the volume density at, and the linear velocity of, a point P_f in a free charge that at any instant occupies a point P in the surface S. Let ρ_b and $\mathbf{u}_b$ be the corresponding quantities at a point P_b in a bound charge that at this instant is undergoing a varying linear displacement within a molecule intersected by this surface, and let ρ_r and $\mathbf{u}_r$ be the corresponding quantities at a point in a bound charge that at this instant has a rotary motion within a molecule intersected by this surface. Then Eq. (E.043) may be written

(E.044) $$\frac{\mathbf{B}}{\mu_0} \cdot \mathbf{ds} = \int_S (\rho_f\mathbf{u}_f) \cdot \mathbf{dS} + \int_S (\rho_r\mathbf{u}_r) \ \mathbf{dS}$$

$$\frac{\partial}{\partial t}\Big) \cdot \mathbf{dS} \quad \text{amperes}$$

The integrand in the first term in the right-hand member of this expression is the density $\mathbf{J}$ of the *conduction current* at any point P, *viz.*,

(E.045) $$\rho_f \mathbf{u}_f = \mathbf{J} \quad \text{amperes per square meter}$$

The integrand in the third member of this expression may be expressed as the time rate of increase of a vector $\mathbf{D}$, *viz.*,

(E.046) $$\rho_b \mathbf{u}_b + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{\partial \mathbf{D}}{\partial t}$$
$$\text{amperes per square meter}$$

From the macroscopic point of view this vector $\mathbf{D}$ is identical with the electrostatic flux density, *viz.*,

(E.047) $$\mathbf{D} = \epsilon \mathbf{E} \quad \text{coulombs per square meter}$$

where ϵ is the permittivity of the medium at the point under consideration. If ϵ is a constant, it follows that

(E.048) $$\frac{\partial \mathbf{D}}{\partial t} : \frac{\partial \mathbf{E}}{\partial t}$$

Designate by $\mathbf{M}$ the vector whose *curl* is the product of μ_0 by the vector $(\rho_r \mathbf{u}_r)$, *viz.*,

(E.049) $$\rho_r \mathbf{u}_r = \frac{1}{\mu_0} \boldsymbol{\nabla} \times \mathbf{M} \quad \text{amperes per square meter}$$

Then, from Stokes's theorem, Eq. (E.030), the second integral in the right-hand member of Eq. (E.45) may be written

(E.050) $$\int_S (\rho_r \mathbf{u}_r) \cdot \mathbf{dS} = \int_C \frac{\mathbf{M}}{\mu_0} \cdot \mathbf{ds} \quad \text{amperes}$$

Designate by $\mathbf{H}$ the vector

(E.051) $$\mathbf{H} = \frac{\mathbf{B} - \mathbf{M}}{\mu_0} \quad \text{amperes per meter}$$

Then Eq. (E.045) may be written

(E.052) $$\frac{\partial \cdots}{\partial \cdots}) \cdot \mathbf{dS} \quad \text{amperes}$$

From the macroscopic point of view the vector $\mathbf{M}$ is the *magnetic polarization* of the medium at the point under con-

sideration and the vector **H** is the *magnetic field intensity*, or *magnetizing force*, at this point. This magnetizing force may also be written

(E.053)   amperes per meter

where by definition μ is the *magnetic permeability* of the medium at the point under consideration. By combining Eqs. (E.051) and (E.053) it is seen that the magnetic polarization **M** may also be written, from the macroscopic point of view,

(E.054) $\mathbf{M} = \mathbf{B} - \mu_0\mathbf{H} = (\mu - \mu_0)\mathbf{H}$ webers per square meter

The vector $\partial\mathbf{D}/\partial t$ is usually referred to as the *density of the displacement current* in a dielectric, and its flux through a given surface, *viz.*,

(E.055) $\dfrac{\partial\psi}{\partial t} = \dfrac{\partial}{\partial t}$ amperes

as the *displacement current* through this surface. The reader should note, however, that through a surface in free space, where $\mathbf{D} = \epsilon_0\mathbf{E}$, there is no actual flow of electricity, although this so-called *displacement current* does have a value, its density at any point being $\epsilon_0\partial\mathbf{E}/\partial t$.

From Stokes's theorem Eq. (E.052) may also be written

(E.056) $\dfrac{\partial}{\partial t}$ amperes per square meter

Compare with Eq. (E.041) for the curl of the electric-field intensity **E**.

E.06. Ampère's Law.—That portion $\mathbf{B}_c$ of the magnetic-flux density which is attributable solely to the *conduction currents* in the field, neglecting both magnetic and electric polarization, is the curl of the vector potential

(E.057) $\dfrac{\mathbf{J}_c\,dv}{r}$ webers per meter

where $\mathbf{J}_c = \rho_f\mathbf{u}_f$ is the density of the conduction current at any point. This flux density $\mathbf{B}_c$ is then

(E.058) $\mathbf{B}_c = \boldsymbol{\nabla} \times \mathbf{A}_c$ webers per square meter

and the corresponding magnetizing force

(E.059)
$$\mathbf{H}_c = \frac{\mathbf{B}_c}{\mu_0} \qquad \text{amperes per meter}$$

is conveniently referred to as the magnetizing force attributable solely to the conduction currents, magnetic and electric polarization being neglected.   This magnetizing force $\mathbf{H}_c$ may also be written

(E.060)
$$\mathbf{H}_c = \frac{1}{\mu_0} (\qquad \mathbf{A}_c) \qquad \text{amperes per meter}$$

From the vector potential $\mathbf{A}_c$ defined by Eq. (E.057) may be deduced an extremely useful relation known as *Ampère's law*.
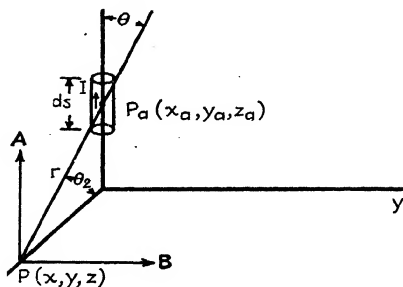
Fig. E.02.

Consider a volume v which is a right cylinder (for example, a straight portion of a conducting wire), which has a cross section S and a length ds, and in which the direction of flow of the conduction current I is parallel to the axis of this cylinder.   Under these conditions, when r is large compared to the linear dimensions of the volume of the cylinder, the vector potential at any point P at a distance r from this cylinder of infinitesimal length ds has the magnitude

(E.061)*
$$\mathbf{A}_c^* = \frac{\mu_0}{4\pi} : \qquad \text{amperes}$$

and is in the direction of the positive sense of I.   The curl of

* The asterisk is used in the symbol $\mathbf{A}_c^*$ and in the symbols for the components of $\mathbf{B}_c^*$ and $\mathbf{H}_c^*$ *to indicate that these quantities are differentials*.   Thus $\mathbf{A}_c^*/ds$ is the vector potential *per unit length of the conductor*.

this vector at any point P which is at a distance r from the center P_a of this cylinder may be formulated as follows.   Choose as the y axis a line that is parallel to the axis of this cylinder and is drawn in the positive sense of the current I (see Fig. E.02). Choose as the yz plane any plane through this axis and choose as the x axis a line perpendicular to this plane in the sense that makes the three mutually perpendicular axes a right-handed system.   From Eq. (E.061) the vector potential $\mathbf{A}_c^*$ at the point P is in the direction of a line parallel to and in the positive sense of the y axis, and this is true for *all* positions of the point P that are not too close to P_a.   Therefore

(E.062)
$$A_{cx}^* = 0$$
$$\frac{\mu_0 I\ ds}{4\pi} \cdot \frac{1}{r}$$
$$A_{cz}^* = 0$$

where for any point P

(E.063) $r = \sqrt{(x - x_a)^2 + (y - y_a)^2 + (z - z_a)^2}$

Since Eqs. (E.061) and (E.062) hold for all values of x, y, and z when r is large compared with the linear dimensions of the cylinder, the curl of the vector $\mathbf{A}_c^*$ defined by Eq. (E.061) is the vector $\mathbf{B}_c^*$ whose three axial components are respectively

$$\partial A_c^{\ *}$$

(E.064) $$\partial A_{cx}^* \qquad \partial A_c^{\ *}$$

$$\partial x \qquad \partial y \qquad\qquad \cos \theta_x$$

where θ_z is the angle made by the line from P_a to P with the positive sense of the z axis and θ_x is the angle made by this line with the x axis.   But θ_x is $\pi/2$ so that $\cos \theta_x$ is zero, and

(E.065) $\cos \theta_z = \sin \theta$

where θ is the angle between the positive sense of the current I in the cylinder and the direction of the line from the cylinder to the point P.   Hence the magnetic-flux density at the point

P has the magnitude

(E.066)
$$\frac{\sin \theta) \, ds}{4\pi r^2}$$

and its direction is that of the line drawn through P perpendicular to the plane formed by the axis of the cylinder and the line from the center of the cylinder to P, in the sense indicated by the arrow marked **B** in Fig. E.02. The vector which has this same direction but the magnitude

(E.067)
$$= \frac{B_c^*}{\mu_0} = \frac{(I \sin \theta) \, ds}{}$$

is defined as the *magnetizing force* at the point P *due solely to the conduction current* in the cylinder of length ds.

Since the relation just deduced holds for every value of r, it must hold for all points on a circle that lies in a plane that has its center on the axis (extended) of the cylinder a fixed distance from the center of the cylinder. Therefore, for all points on such a circle the magnetizing force $\mathbf{H}_c$ has the value given by Eq. (E.067), and its direction at any point is the direction in which a point moves in tracing this circle in the right-handed-screw direction with respect to the positive sense of the current in the cylinder. Equation (E.067) and this rule regarding the direction of the magnetizing force $\mathbf{H}_c$ due to a conduction current in a cylinder, at a distance r from this cylinder, is known as *Ampère's law.*

By applying Eq. (E.067) to *all* the successive macroscopically infinitesimal elements that make up one or more *closed* paths in which the conduction currents are $I_1, I_2, \ldots , I_n$, and adding the results vectorially, one may obtain the total magnetizing force $\mathbf{H}_c$ *due solely to these conduction currents*. In such a formulation the effect of magnetic polarization and the effect of the time variation of the electrostatic-flux density in the surrounding medium are both neglected.

E.07. Magnetic Poles and Magnetic Scalar Potential.—To obtain at any point P the total magnetizing force **H** one must add vectorially to the magnetizing force $\mathbf{H}_c$ due to the conduction currents the magnetizing force $\mathbf{H}_m$ attributable to the magnetic polarization **M** and to the varying electric polarization $\partial \mathbf{D}/\partial t$ which may exist in the magnetic and dielectric media in the field. When the effect of the varying electric polarization is negligible, as

is usually the case unless the time variation of **D** is very rapid, one need consider only the additional magnetizing force attributable to the magnetic polarization. This may be expressed in terms of a scalar m associated with each contact layer between a magnetic and a nonmagnetic surface or between any two contact surfaces of *different* magnetic permeabilities.

The procedure is as follows. Since the magnetic flux through *every* surface bounded by a closed loop C has the same value, irrespective of whether this surface is a contact surface or not, the normal components $B_1 \cos \theta_1$ and $B_2 \cos \theta_2$ of the flux densities at two points on the normal to a contact surface, one in medium No. 1 and the other in medium No. 2 must be equal, *viz.*,

$$\text{(E.068)} \qquad \qquad \cos \qquad = B_2 \cos \theta_2$$

From Eq. (E.052), when there is no conduction or displacement current in the contact layer between the two media, the tangential components $H_1 \sin \theta_1$ and $H_2 \sin \theta_2$ of the total magnetizing force at these two points must likewise be equal, *viz.*,

$$\text{(E.069)} \qquad \qquad H_1 \sin \theta_1 = H_2 \sin \theta_2$$

Incidentally, therefore, the ratios of the angles, θ_1 and θ_2 made by both **B** and **H** with the normal to this contact surface bear to each other the relation

$$\text{(E.070)} \qquad \qquad \frac{\tan}{\tan \theta_2} \qquad B_1 H_1 \qquad \mu_1$$

Compare with Eq. (10.119).

Form the scalar

$$\text{(E.071)} \qquad m = \mu_0 \int_S (H_2 \cos \theta_2 - H_1 \cos \theta_1) \, dS \qquad \text{webers}$$

where S is the contact surface. This scalar is equal to the product of μ_0 by the total outward flux of the vector **H** from the two faces of the contact layer between the two media. From Eq. (E.068) and the relations

$$\text{(E.072)} \qquad \qquad H_1 = \frac{B_1}{\mu}$$

$$\text{(E.073)} \qquad \qquad \qquad \mu_2$$

Eq. (E.071) may be written

$$(E.074) \qquad m = \left(\frac{\mu_0}{} - \frac{\mu_0}{\mu} \right) \qquad \text{webers}$$

where ϕ_{12} is the total flux through the contact surface S. This scalar m is called the magnetic pole strength at the surface S and is said to be a *north pole* when m is positive and a *south pole* when m is negative.

Designate by ρ_m the surface density of this scalar at any point (macroscopic) in the contact surface S, *viz.*,

$$(E.075) \qquad \rho_m = \frac{dm}{dS} = \left(\frac{\mu_0}{\mu_2} - \frac{\mu_0}{\mu_1} \right) \frac{d\phi}{dS} \qquad \text{webers per square meter}$$

where $d\phi$ is the magnetic flux through the macroscopically infinitesimal area dS. Form the *scalar potential*

$$(E.076) \qquad U = \int_S \frac{\rho_m \, dS}{r} \qquad \text{amperes}$$

where r is the distance of a specified point P from dS, and the integral includes every contact surface at which ρ_m has a value different from zero. This scalar U is called the *magnetic scalar potential* at the point P. The observed phenomena are therefore in accord with the assumption that, at the point P, the contribution $\mathbf{H}_m$ to the magnetizing force attributable to all the magnetically polarized bodies in the field is the *negative of the gradient* of this magnetic scalar potential, *viz.*,

$$(E.077) \qquad \mathbf{H}_m = -\boldsymbol{\nabla} U \qquad \text{amperes per meter}$$

The total magnetizing force at the point P attributable to *all* the conduction currents in the field and all the magnetically polarized bodies in the field (all displacement currents being neglected) is then the vector sum of

$$(E.078) \qquad \mathbf{H}_c = \boldsymbol{\nabla} \times \mathbf{A}_c \qquad \text{amperes per meter}$$

where $\mathbf{A}_c$ is the modified vector potential, and $\mathbf{H}_m$ as calculated from Eq. (E.077), *viz.*,

$$(E.079) \qquad \mathbf{H} = \mathbf{H}_c + \mathbf{H}_m \qquad \text{amperes per meter}$$

The corresponding value of the magnetic-flux density at the

point P is then

(E.080) $\mathbf{B} = $ webers per square meter

where μ is the *magnetic permeability at this point P*.

The method just outlined for calculating the magnetizing force $\mathbf{H}_m$ attributable to the magnetic polarization of one or more bodies in a magnetic field is seldom applicable, since the pole strength m at any surface depends upon the value of the surface integral of $\mu_0\mathbf{H}$ over the two sides of this surface, which in turn depends upon the value of $\mathbf{H}_m$ at each point of these two sides. However, the method is sometimes useful when the distribution of the magnetic poles can be estimated from other considerations. Also note that when two poles m and m' may be considered as concentrated in two points at a distance r apart then the mechanical force of repulsion exerted by one pole on the other is

(E.081) $F = \dfrac{mm'}{}$ newtons

Compare with Coulomb's law for the force of repulsion between two point charges.

E.08. The Self-inductance and Mutual Inductance of Parallel Wires.—A simple example of the use of the vector potential $\mathbf{A}$ is the calculation of the self-inductance per unit length of a very long right cylinder of circular cross section and of the mutual inductance per unit length between two such cylinders when they are parallel. Note first of all that, if the vector potential at each point of a straight line of unit length has the magnitude A and is in the direction in which this line is drawn, then from Eq. (E.038) the electromotional intensity at each point of this line has the magnitude

(E.082) $E'' = -\dfrac{\partial A}{\partial t}$ volts per meter

and is in the direction of this line. Therefore the emf induced in this unit length by the currents that give rise to this vector potential is identical with E'', *viz.*,

(E.083) $\dfrac{\partial A}{\partial t}$ volts per meter

Consider first the case of a single lineal filament of length 2λ and cross section dS and assume the permeability of both this filament and the surrounding medium to be μ_0. Let J be the magnitude of the conduction-current density at each point of this filament and let this current density have a direction parallel to the axis of this filament in a specified sense. Then the current in this sense is

$$(\text{E.084}) \qquad\qquad dI = J \, dS \qquad\qquad \text{amperes}$$

Choose as the z axis a line parallel to the axis of this filament in this specified sense and choose as the xy plane the plane that intersects this axis at its middle point. Let P be a point in the xy plane at a normal distance N from this axis and at a distance $\sqrt{z^2 + N^2}$ from a point on this axis whose z coordinate is z.

Since, by hypothesis, the magnetic permeability of the filament and the surrounding medium is the constant μ_0, there will be no magnetic polarization at any point in the magnetic field produced by this current. Also assume that there is no *varying* electric polarization at any point in the filament or in the surrounding medium. Then the vector potential at P due to the current in this filament has the magnitude

$$(\text{E.085}) \qquad\qquad dA = \int_0^\lambda \quad dz$$

and its direction is the same as the positive sense of the z axis. The integral in Eq. (E.084) is identical with the last integral in Eq. (D.02). Therefore all the deductions in Sec. D.02 relative to the *scalar* potential V due to a charged filament, cylindrical shell, and solid cylinder also hold for the *vector* potential due to a conduction current in the corresponding shapes, provided Q is changed to I and ϵ_0 to $1/\mu_0$.

The only case that will be considered here is that of two or more parallel solid circular cylinders (round wires) in each of which the conduction current has a uniform current density and all of which have their ends in two parallel planes a distance 2λ apart. It will also be assumed that the distance between the axes of any two of these conductors is small compared with 2λ and large compared with the radii of these cylinders. Designate by r_n the normal distance of any point P from the axis of the nth

cylinder in this group. Designate by R_n the radius of this cylinder and by I_n the current in it. The resultant vector potential **A** at this point P is in the direction of the positive sense of the z axis, and that part of this vector potential which is attributable to the current I_n has the magnitude

(E.086) $A_{on} =$ for $r_n > R_n$

(E.087)

Compare with Eqs. (D.28) and (D.33). Also note that the *average* value of A_{in} over the entire cross section of the nth cylinder is

(E.088) $A_{in} = \dfrac{\mu_0 I_n}{2\pi} \left[\log\left(\dfrac{2\lambda}{R_n}\right) + \dfrac{1}{4} \right]$ webers per meter

Compare with Eq. (D.35).

From Eqs. (E.083) and (E.088) the average value of the emf induced per unit length in any one of the wires, say wire No. 1, as a consequence of the time variation of the current I_1 in this wire is

(E.089) $= -\dfrac{\mu_0}{2\pi} \left\lfloor \log\left(\dfrac{2\lambda}{R_1}\right) + \dfrac{1}{4} \right\rfloor \dfrac{dI_1}{dt}$ volts per meter

By definition the factor which multiplies $-dI_1/dt$ in this expression is the self-inductance L_{11} of this wire per unit length, *viz.*,

(E.090) $L_{11} = \dfrac{\mu_0}{2\pi} \left[\log\left(\dfrac{2\lambda}{R_1}\right) + \dfrac{1}{4} \right]$ henrys per meter

From Eqs. (E.083) and (E.086) the average value of the emf induced per unit length in wire No. 1 as a consequence of the time variation of the current I_2 in wire No. 2 whose axis is at a normal distance D_{12} from wire No. 1 is

(E.091) $e_{12} = -\dfrac{\mu_0}{2\pi} \log\left(\dfrac{2\lambda}{D_{12}}\right) \dfrac{dI_2}{dt}$ volts per meter

By definition the factor which multiplies $-dI_2/dt$ in this expression is the mutual inductance L_{12} of wire No. 2 and unit length

of wire No. 1, *viz.*,

(E.092) $$L_{12} = \frac{\mu_0}{2\pi} \log \frac{2\lambda}{R} \qquad \text{henrys per meter}$$

When I_1 and I_2 are equal in magnitude but opposite in sign then the total average emf induced per unit length in wire No. 1 by *both* of these currents is

(E.093) $$= e_1 - e_2 = -L_1$$

where L_1 is the difference between L_{11} and L_{12}, *viz.*,

(E.094) $$L_1 = \frac{\mu_0}{2\pi} \left[\log \left(\frac{D_{12}}{R_1} \right) + \frac{1}{4} \right] \quad \text{henrys per meter of wire}$$

This quantity L_1 is the effective self-inductance per unit length of wire No. 1 and is the value usually given in the tables of self-inductance in handbooks.

From the analysis above given it is obvious that Eqs. (E.090), (E.092), and (E.094) are all approximate formulas, since they are based on the *average* values of the emfs induced in the lineal filaments that make up the wires and also upon the assumption of uniform current distribution throughout each wire. Actually, the back emfs in the interior filaments of a wire, caused by a varying current in that wire, are greater than the emfs in those near the surface, with the result that, to maintain each cross section of the wire an equipotential surface, the current density must be greater near the surface than in the interior of the wire. This effect is the so-called *skin effect* (see Sec. 13.06).

Also note that when the two wires are close together, *i.e.*, when R_1 and R_2 are small compared with D_{12}, then the back emf due to mutual induction in those filaments which are at a distance apart greater than D_{12} is greater than the back emf in those filaments which are at a distance apart less than D_{12}. This causes the current density to tend to a greater value in those parts of the two wires which are nearest together than in those parts which are farther apart. This so-called *proximity effect* is analogous to the non-uniform distribution of electric charge on two parallel wires that are close together (see Sec. D.05).

Another item to be noted is that if a wire of circular cross section has a permeability μ, instead of μ_0, then the $\frac{1}{4}$ in Eqs. (E.090) and (E.092) becomes $\mu/4\mu_0$ (see Sec. E.09).

By making use of Eqs. (E.090) and (E.092) one can readily determine the effective self- and mutual inductance when there are currents in more than two parallel wires, provided the algebraic sum of these currents in a specified sense is zero.

E.09. Vector Potential When Magnetic and Electric Polarization Are Present.—The basic formula for vector potential is

(E.095) $$\qquad\qquad (\rho\mathbf{u})\ dv \qquad$$ webers per meter

in which the integration refers to *all* the moving charges in the field. As in Eq. (E.045) this integral may be considered as the vector sum of three integrals, *viz.*,

(E.096) $$\mathbf{A} = \mathbf{A}_c + \mathbf{A}_m + \mathbf{A}_p$$

In this expression the vector

(E.097) $$\qquad\qquad \frac{\mathbf{J}_c\ dv}{r} \qquad$$ webers per meter

is the contribution to the resultant vector potential attributable to the conduction currents, the density of which at any point is

(E.098) $$\mathbf{J}_c = \rho_f \mathbf{u}_f \qquad$$ amperes per square meter

The vector

(E.099) $$\qquad\qquad \frac{\mathbf{J}_m\ dv}{4\pi} \qquad$$ webers per meter

is the contribution to the resultant vector potential attributable to the magnetic polarization $\mathbf{M}$, and $\mathbf{J}_m$ has the value at a point

(E.100) $$\mathbf{J}_m = \quad = \frac{1}{\mu_0}\boldsymbol{\nabla}\times\mathbf{M} \qquad$$ amperes per square meter

where ρ_r and $\mathbf{u}_r$ refer to the charges that have a rotary motion within the macroscopically infinitesimal volume surrounding this point. The vector potential

(E.101) $$\mathbf{A}_p = \quad \frac{\mathbf{J}_p\ dv}{r} \qquad$$ webers per meter

is the contribution to the resultant vector potential attributable to the electric polarization **P**, and $\mathbf{J}_p$ has the value

$$(E.102) \qquad\qquad\qquad = \frac{\partial \mathbf{P}}{\partial t} \quad \text{amperes per square meter}$$

where ρ_b and $\mathbf{u}_b$ refer to the charges that undergo varying linear displacements within any macroscopically infinitesimal volume. Note that $\mathbf{J}_p$ is not the density of the entire displacement current but only the density of that part of the displacement which is due to the electric polarization; for example, $\mathbf{J}_p$ is zero wherever $\epsilon = \epsilon_0$, whereas the displacement current density at such a point is $\epsilon_0 \, \partial\mathbf{E}/\partial t$.

When there is no magnetic or electric polarization, *i.e.*, when $\epsilon = \epsilon_0$ and $\mu = \mu_0$ at every point in the field, the two components $\mathbf{A}_m$ and $\mathbf{A}_p$ of the vector potential **A** do not exist, and therefore $\mathbf{A}_c$ is identical with the resultant vector potential **A**. To calculate $\mathbf{A}_m$ one must know the magnetic polarization at every point, which in turn depends in general upon the resultant vector potential **A**. Similarly, to calculate $\mathbf{A}_p$ one must know the electric polarization at every point, which in general depends upon both the resultant vector potential **A** and the scalar potential *V*. Consequently Eqs. (E.099) and (E.101) are, as a rule, of no particular value for actual computations, but they are important in that they coordinate the effects of the motions of electric charges (both whorls and linear displacements) which are confined to individual macroscopic volumes with the effect of conduction currents (drifting charges).

One application of Eq. (E.099) is to the calculation of the contribution to the resultant vector potential of the magnetic polarization produced in a right circular cylinder of length 2λ and radius R by a conduction current I, uniformly distributed throughout this cylinder. Note first that at any point P_i in this cylinder, at a normal distance r_i from its axis, the vector potential $\mathbf{A}_c$ attributable directly to the conduction current I in the cylinder has, from Eq. (E.087), the magnitude

$$(E.103) \qquad\qquad = \frac{\mu_0 I}{2\pi}\left(\log\frac{2\lambda}{R} + \frac{R^2 - r_i^2}{2R^2}\right) \quad \text{webers per meter}$$

and is in the positive sense of the z axis. The curl of this quantity gives the magnetic-flux density B_c at this point attributable

directly to the conduction current I. The axial components of this vector potential $\mathbf{A}_c$ at all points within the given cylinder are

(E.104)
$$A_{cz} = \frac{\mu_0 I}{2\pi} \qquad \frac{2\lambda}{R}$$

Also note that in the last of these expressions the only space variable is

(E.105) $r^2 = x^2 + y^2$

where x and y are the coordinates of the point P_i. Hence the components of the curl of $\mathbf{A}_c$, *viz.*, the components of the magnetic-flux density $\mathbf{B}_c$ attributable directly to the current I are

(E.106)
$$\partial y \qquad \frac{\partial A_{cy}}{\partial z} =$$
$$\partial z \qquad \frac{\partial A_{cz}}{\partial x}$$
$$\partial x \qquad \frac{\partial A_{cz}}{\partial y} = 0$$

Therefore the magnitude of $\mathbf{B}_c$ at any point P_i within this cylinder, at a normal distance r from its axis, is

(E.107) $B_c =$ webers per square meter

and the direction of this flux density is tangent to the circle through P_i which has its center on the axis of the cylinder, and the sense of this flux density is a right-handed-screw direction with respect to the sense of the current I.

The flux lines of this vector $\mathbf{B}_c$ are therefore circles concentric with the axis of the cylinder, and so are the flux lines of the magnetizing force

(E.108) $\dfrac{I r}{\mu_0}$ amperes per meter

attributable directly to this current. Consequently if the cylinder is a homogeneous isotropic material of permeability μ, the magnetic polarization $\mathbf{M}$ therein must be a vector whose flux

lines are likewise circles concentric with the axis of this cylinder. In this case there are no magnetic poles, and therefore $\mathbf{H}_c$ is the resultant magnetizing force at the point P_i. Hence from Eq. (E.054) the magnetic polarization at a point P_i at a normal distance $r_i \lessgtr R$ from the axis of the cylinder has the same direction as H_c and B_c at this point and has the magnitude

(E.109) $$M = (\mu - \mu_0)H_c = Kr_i \quad \text{webers per square meter}$$

where K has the value

(E.110) $$= \frac{(\mu - \mu_0)I}{}$$

This quantity K is a constant relative to the coordinates of the point P_i. When the medium outside this cylinder is non-magnetic, *i.e.*, when the permeability outside this cylinder is μ_0, then the polarization at every point in the surrounding medium is *zero*.

Designate by x and y the coordinates of the point P_i in the xy plane. Then the three components of the polarization $\mathbf{M}$ inside the cylinder are

(E.111) $$\begin{aligned} M_x &= -Ky_a \\ M_y &= Kx_a \\ M_z &= 0 \end{aligned}$$

Hence at any point P inside the cylinder, *but not in the contact surface between it and the surrounding medium*, the curl of the polarization has the components

(E.112) $$\begin{aligned} (\nabla \times \mathbf{M}) & \\ (\nabla \times \mathbf{M})_y &= 0 \\ (\nabla \times \mathbf{M})_z &= 2K \end{aligned}$$

For all points inside the cylinder, but not in the contact surface between it and the surrounding medium, the vector $\mathbf{J}_m$ therefore has the magnitude

$$\mu_0 \qquad\qquad \mu_0 \qquad \text{amperes per square meter}$$

and is in the *positive* sense of the z axis.

When the boundary between the cylinder of permeability μ and the surrounding medium of permeability μ_0 is considered as a geometric surface (zero thickness) then the interpretation of

$(\nabla \times \mathbf{M})$ as the cross product of the vector operator ∇ and the vector $\mathbf{M}$ gives for $(\nabla \times \mathbf{M})$ an *infinity*. However, if this contact surface is considered as a contact *layer* that has a macroscopically infinitisimal thickness dw, in which $\mathbf{M}$ changes continuously from KR to zero, then from Stokes's theorems the *average* value of $(\nabla \times \mathbf{M})$ in this contact layer has the magnitude

where the closed loop C is the "rectangle" of width dw and length $Rd\theta$ around a point P_s in the circle of radius R (see Fig. E.03)
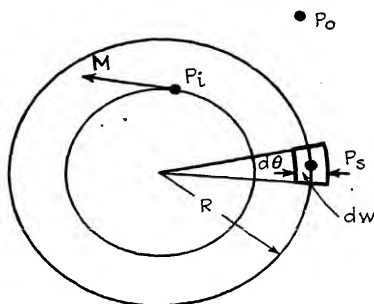
Fig. E.03.

Since $\mathbf{M}$ has no component along the radial ends of this loop and is zero along the outside edge, and along the inside edge of this rectangle has the magnitude

(E.114) $M_R = KR$

and has the sense indicated by the arrow, the magnitude of $(\nabla \times \mathbf{M})_{av}$ is

(E.115) $(\nabla \times \mathbf{M})_{av} = \dfrac{M_R}{dw} \qquad \dfrac{KR}{dw}$

and this vector $(\nabla \times \mathbf{M})$ is in the direction of the *negative* sense of the z axis.

From the above analysis it follows that the vector potential at any point inside or outside a cylinder of radius R, attributable to the polarization of this cylinder by the current I, is in the positive sense of the z axis and has a magnitude that is the *difference* between the vector potential $\mathbf{A}'_m$ which would be pro-

duced by a conduction current I_m' of density $2K/\mu_0$ at each point of the cross section of this cylinder were its permeability μ_0, and the vector potential $\mathbf{A}_m''$ per unit length which would be produced by a conduction current I_m'' of density $KR/\mu_0 \, dw$ in the shell of radius R and thickness dw which forms the contact surface between this cylinder and the surrounding medium. Since the area of the cross section of the cylinder is πR^2 and the area of the cross section of the shell of radius R and thickness dw is $2\pi R \, dw$, it follows that these two currents I_m' and I_m'' have the *same* magnitudes, *viz.*,

$$(\text{E.116}) \qquad I_m' = I_m'' = \qquad \frac{}{\mu_0} \qquad \frac{-\ \mu_0}{\mu_0}.$$

Since the two equivalent currents I_m' and I_m'' are parallel to the axis of the cylinder and are in *opposite directions*, and since with respect to the vector potential at any point P_o outside the cylinder each may be considered as concentrated at this axis, their combined effect on the vector potential at P_o is nil. Hence the total vector potential at P_o due to the actual current I, in the cylinder of permeability μ is the same as would be produced by this same actual current in this cylinder were its permeability μ_0.

For any point P_i inside the cylinder the vector potential, in the positive sense of the z axis, produced by the equivalent current I_m' in the solid cylinder is, from Eq. (E.103)

and the vector potential in this same sense produced by the equivalent current I_m'' in the shell formed by the surface layer is, from Eq. (D.23) when dQ is changed to I_m'' and $1/\epsilon_0$ to μ_0

$$(\text{E.118}) \qquad A_m'' = \ -$$

Therefore the magnitude of the resultant vector potential at P_i due to the polarization of the cylinder, *viz.*, the sum of A_m' and

$$A_n \qquad \frac{}{2\pi} \qquad \frac{-\ r_i^2}{} \bigg) = \frac{(\mu\ -}{}$$

The sum of this vector potential A_m and the vector potential *inside* the cylinder due to the actual current I, which latter potential is given by Eq. (E.103), is

The average value of this vector potential inside the cylinder of permeability μ is therefore

(E.121) $$A_{ia} = \frac{\mu_0 I}{\Omega} \qquad \frac{2\lambda}{R} \qquad \overline{4} \ \overline{\mu_0}$$

Therefore, as noted in Sec. E.08, the effective self-inductance per unit length of such a cylinder, due to the current in it and a return current $-I$ in another cylinder, when the axes of these cylinders are separated by a normal distance D, is

(E.122) $$L = \frac{\mu_0}{2\pi}\left[\log\left(\frac{D}{R}\right) + \frac{1}{4}\frac{\mu}{\mu_0}\right] \quad \text{henrys per meter of wire}$$

Compare with Eq. (E.094).

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